15.1. Let G be a locally compact group. As was shown in the lectures, $L^1(G)$ is a Banach algebra under convolution.
(a) Show that $L^1(G)$ is a Banach $*$-algebra w.r.t. the involution $f^*(x) = \overline{f(x^{-1})}\Delta(x^{-1})$ ($f \in L^1(G)$, $x \in G$).
(b) Show that $L^1(G)$ (equipped with the standard L^1-norm and with the involution defined in (a)) is not a C^*-algebra unless $G = \{e\}$.
(c) Show that $L^1(G)$ is commutative if and only if G is commutative.
(d) Show that $L^1(G)$ is unital if and only if G is discrete.
(e) Show that the Fourier transform $\mathscr{F}\colon L^1(G) \to \big(\bigoplus_{\sigma\in\widehat{G}} \mathscr{B}(E^\sigma)\big)_\infty$ is a $*$-algebra homomorphism.

15.2. Let G be a locally compact group, and let $p, q \in (1, +\infty)$ satisfy $1/p + 1/q = 1$. Show that, for each $f \in L^p(G)$ and $g \in L^q(G)$, the convolution $f * Sg$ (where $(Sg)(x) = g(x^{-1})$) is defined everywhere on G, belongs to $C_0(G)$, and that $\|f * Sg\|_\infty \leqslant \|f\|_p\|g\|_q$.

15.3. (In this exercise, we use the notation introduced in Sheet 13.) Let G be a compact group. For each $\sigma \in \widehat{G}$, let $p_\sigma = d_\sigma\chi_\sigma$, where $d_\sigma = \dim E^\sigma$.
(a) Calculate $\pi^\sigma_S * \pi^\tau_T$ (where $S \in \mathrm{End}(E^\sigma)$, $T \in \mathrm{End}(E^\tau)$). Calculate $p_\sigma * p_\tau$.
(b) Show that $\mathscr{R}_\sigma(G)$ is a two-sided ideal of $L^1(G)$, and that $\mathscr{R}_\sigma(G)$ is $*$-isomorphic to the matrix algebra $M_{d_\sigma}(\mathbb{C})$. What is the identity of $\mathscr{R}_\sigma(G)$?

15.4. Show that **(a)** $C^n[a, b]$ ($n \geqslant 1$) and **(b)** $\mathscr{A}(\bar{\mathbb{D}})$ are Banach $*$-algebras, but are not C^*-algebras. (Recall that the involution on $C^n[a, b]$ is given by $f^*(t) = \overline{f(t)}$, while the involution on $\mathscr{A}(\bar{\mathbb{D}})$ is given by $f^*(z) = \overline{f(\bar{z})}$.)

15.5. Let A be a normed algebra, and let (e_α) be a bounded approximate identity in A. Show that
(a) if B is a normed algebra and $\varphi\colon A \to B$ is a continuous homomorphism such that $\overline{\varphi(A)} = B$, then $(\varphi(e_\alpha))$ is a bounded approximate identity in B;
(b) if A is a normed $*$-algebra, then $(e^*_\alpha e_\alpha)$ is a bounded approximate identity in A.

15.6. Let X be a locally compact Hausdorff topological space.
(a) Construct a bounded approximate identity in $C_0(X)$.
(b) Show that $C_0(X)$ has a sequential bounded approximate identity if and only if X is σ-compact.

15.7. Let H be a Hilbert space.
(a) Construct a bounded approximate identity in $\mathscr{K}(H)$.
(b) Show that $\mathscr{K}(H)$ has a sequential bounded approximate identity if and only if H is separable.

15.8. Let G be a locally compact group, and let (u_i) be a Dirac net in $L^1(G)$. Identify $L^1(G)$ with a subspace of $C_0(G)^*$ via $f \mapsto I_f$, where $I_f(g) = \int_G fg\,d\mu$ ($g \in C_0(G)$). Show that (u_i) converges to the evaluation functional $g \mapsto g(e)$ (the "Dirac δ-function") w.r.t. the weak* topology on $C_0(G)^*$.

15.9. Let $\mathscr{P}(\mathbb{T})$ denote the closure of $\mathbb{C}[z]$ in $C(\mathbb{T})$, where z is the coordinate on $\mathbb{C}$. Recall that the *disk algebra* $\mathscr{A}(\bar{\mathbb{D}})$ consists of those $f \in C(\bar{\mathbb{D}})$ that are holomorphic on the disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Show that each $f \in \mathscr{P}(\mathbb{T})$ uniquely extends to $\tilde{f} \in \mathscr{A}(\bar{\mathbb{D}})$, and that $\sigma_{\mathscr{P}(\mathbb{T})}(f) = \tilde{f}(\bar{\mathbb{D}})$.

15.10. Let $c_{00} \subset c_0$ denote the ideal of finite sequences (i.e., of those sequences $a = (a_n)$ such that $a_n = 0$ for all but finitely many $n \in \mathbb{N}$). Prove that c_{00} is not contained in a maximal ideal of c_0.

15.11. Let $A = \{f \in C[0, 1] : f(0) = 0\}$, and let $I = \{f \in A : f$ vanishes on a neighborhood of $0\}$. Prove that I is not contained in a maximal ideal of A.

15.12. A commutative algebra A is *semisimple* if the intersection of all maximal modular ideals of A (the *Jacobson radical* of A) is $\{0\}$. Show that every homomorphism from a Banach algebra to a commutative semisimple Banach algebra is continuous.

15.13. Describe the maximal spectrum and the Gelfand transform for the algebras **(a)** $C^n[0,1]$; **(b)** $\mathscr{A}(\bar{\mathbb{D}})$; **(c)** $\mathscr{P}(\mathbb{T})$; **(d)** $\ell^1(\mathbb{Z})$.

15.14. Let $A(\mathbb{T}) = \{f \in C(\mathbb{T}) : \sum_{n\in\mathbb{Z}} |\hat{f}(n)| < \infty\}$, where $\hat{f}(n)$ is the nth Fourier coefficient of f w.r.t. the trigonometric system (e_n) on $\mathbb{T}$ (i.e., $e_n(z) = z^n$ for all $z \in \mathbb{T}$ and $n \in \mathbb{Z}$). Prove that for each $f \in A(\mathbb{T})$ we have $\sigma_{A(\mathbb{T})}(f) = f(\mathbb{T})$.

15.15. Construct a commutative Banach algebra A such that for each $t \in [0,1]$ there exists a character χ of A with $\|\chi\| = t$. (Clearly, A cannot be unital, see the lectures.)

15.16. **(a)** Does there exist a norm and an involution on $C^1[a,b]$ making it into a C^*-algebra?
(b) Does there exist a norm and an involution on $\mathscr{A}(\bar{\mathbb{D}})$ making it into a C^*-algebra?
(c) Does there exist a norm and an involution on $\ell^1(\mathbb{Z})$ making it into a C^*-algebra?
Remark. In 2.4 (a,b,c), we do not assume that the new norm is equivalent to the original norm.

15.17. Let X be a locally compact Hausdorff topological space, and let X_+ denote the one-point compactification of X. For each $f \in C_0(X)$, define $f_+ \colon X_+ \to \mathbb{C}$ by $f_+(x) = f(x)$ for $x \in X$ and $f_+(\infty) = 0$. Prove that f_+ is continuous, and that the map $C_0(X)_+ \to C(X_+)$, $f + \lambda 1_+ \mapsto f_+ + \lambda$, is an isometric $*$-isomorphism. (Here we assume that $C_0(X)_+$ is equipped with the canonical C^*-norm extending the supremum norm on $C_0(X)$.)

15.18. Let X be a topological space, let $\beta X = \operatorname{Max} C_b(X)$, and let $\varepsilon \colon X \to \beta X$ take each $x \in X$ to the evaluation map $\varepsilon_x \colon C_b(X) \to \mathbb{C}$ given by $\varepsilon_x(f) = f(x)$.
(a) Prove that $(\beta X, \varepsilon)$ is the Stone-Čech compactification of X (i.e., for each compact Hausdorff topological space and each continuous map $f \colon X \to Y$ there exists a unique continuous map $\tilde{f} \colon \beta X \to Y$ such that $\tilde{f} \circ \varepsilon = f$).
(b) Prove that $\varepsilon(X)$ is dense in βX.
(c) Prove that ε is a homeomorphism onto $\varepsilon(X)$ if and only if X is completely regular.

15.19. Let A and B be C^*-algebras. Show that if B is commutative, then each homomorphism from A to B is a $*$-homomorphism. Does the above result hold without the commutativity assumption?

15.20. Let $A = C^1[0,1]$. **(a)** Is A hermitian? **(b)** Does the identity $\|a\| = r(a)$ hold in A?

15.21. Let $A = \mathscr{A}(\bar{\mathbb{D}})$. **(a)** Is A hermitian? **(b)** Does the identity $\|a\| = r(a)$ hold in A?

15.22. **(a)** Let H be a $*$-module over a Banach $*$-algebra A. Assume that $\operatorname{End}_A(H) = \mathbb{C}\mathbf{1}_H$. Show that H is irreducible.
(b) Does (a) hold if H is a Banach A-module (but is not necessarily a $*$-module)?