Throughout this sheet, G is a compact group equipped with a normalized Haar measure μ. Let $\widehat{G}$ denote the unitary dual of G, i.e., the set of unitary equivalence classes of unitary irreducible representations of G. For each $\sigma \in G$, choose a representation (E^σ, π^σ) in σ. Recall (see the lectures) that $\dim E^\sigma < \infty$ for each σ. The algebra of representative functions on G is denoted by $\mathscr{R}(G)$. For each finite-dimensional continuous representation (E, π) of G and each $T \in \mathrm{End}(E)$, we define $\pi_T \in \mathscr{R}(G)$ by $\pi_T(x) = \mathrm{Tr}(T\pi(x))$. The image of the map $\mathrm{End}(E^\sigma) \to \mathscr{R}(G)$, $T \mapsto \pi^\sigma_T$, is denoted by $\mathscr{R}_\sigma(G)$.

Using the fact that every finite-dimensional continuous representation of G is unitarizable (see the lectures), and combining this with Exercise 11.2 (b), we will often identify $\widehat{G}$ with the set of (algebraic) equivalence classes of finite-dimensional continuous irreducible representations of G.

Definition 13.1. A function $f \in L^2(G)$ is *central* if $f(xyx^{-1}) = f(y)$ for all $x \in G$ and almost all $y \in G$. Note that the equality actually holds for all $x, y \in G$ provided that f is continuous.

The space of all central functions in $L^2(G)$ (resp., in $C(G)$, in $\mathscr{R}(G)$, ...) will be denoted by $ZL^2(G)$ (resp., $ZC(G)$, $Z\mathscr{R}(G)$, ...).

13.1. Interpret $ZL^2(G)$, $ZC(G)$, etc., as the submodule of invariants with respect to a suitable action of G on $L^2(G)$, $C(G)$, etc.

Definition 13.2. Given a finite-dimensional continuous representation (E, π) of G, the *character* of π is a function $\chi_\pi \in \mathscr{R}(G)$ given by $\chi_\pi(x) = \mathrm{Tr}\,\pi(x)$. (In other words, $\chi_\pi = \pi_{\mathbf{1}_E}$, see above.) Observe that $\chi_\pi \in Z\mathscr{R}(G)$.

13.2. Prove that **(a)** $\chi_{\pi\oplus\tau} = \chi_\pi + \chi_\tau$; **(b)** $\chi_{\pi\otimes\tau} = \chi_\pi\chi_\tau$; **(c)** $\chi_{\bar\pi} = \overline{\chi_\pi}$.

For each $\sigma \in \widehat{G}$, let $\chi_\sigma = \chi_{\pi^\sigma}$.

13.3. Show that
(a) $Z\mathscr{R}(G)$ is dense in $ZC(G)$ and in $ZL^2(G)$;
(b) $Z\mathscr{R}_\sigma(G) = \mathbb{C}\chi_\sigma$;
(c) $\{\chi_\sigma : \sigma \in \widehat{G}\}$ is a vector space basis of $Z\mathscr{R}(G)$ and an orthonormal basis of $ZL^2(G)$.

13.4. Let π be a finite-dimensional continuous representation of G, and let $\pi = \bigoplus_\sigma m_\sigma \pi^\sigma$ be the decomposition of π into irreducibles (where $m_\sigma \in \mathbb{Z}_{\geqslant 0}$ is the *multiplicity* of σ in π). Show that
(a) $m_\sigma = \langle \chi_\pi \,|\, \chi_\sigma \rangle$ (the inner product in $L^2(G)$);
(b) $\|\chi_\pi\|_2^2 = \sum m_\sigma^2$;
(c) π is irreducible iff $\|\chi_\pi\|_2 = 1$;
(d) two finite-dimensional continuous representations π and τ are isomorphic iff $\chi_\pi = \chi_\tau$.

13.5. Let $\sigma, \tau \in \widehat{G}$. Show that $\sigma \otimes \tau$ contains the one-dimensional trivial representation iff $\tau \cong \bar\sigma$.

13.6. Let $F \subset \widehat{G}$ be a subset containing the one-dimensional trivial representation, closed under complex conjugation and having the property that, for each $\sigma, \tau \in F$, all irreducible subrepresentations of $\sigma \otimes \tau$ belong to F. Let $\mathscr{R}_F(G) = \sum_{\sigma\in F} \mathscr{R}_\sigma(G)$ (in other words, $\mathscr{R}_F(G)$ is the linear span of matrix elements of representations belonging to F). Show that
(a) $\mathscr{R}_F(G)$ is a unital $*$-subalgebra of $\mathscr{R}(G)$;
(b) if F separates the points of G, then $F = \widehat{G}$.

13.7. Let π be a finite-dimensional continuous representation of G. Prove that the following conditions are equivalent:

(i) π is faithful;
(ii) for each $\sigma \in \widehat{G}$, π^σ is isomorphic to a subrepresentation of $\pi^{\otimes k} \otimes \bar\pi^{\otimes \ell}$ for some $k, \ell \in \mathbb{Z}_{\geqslant 0}$;
(iii) $\mathscr{R}(G)$ is generated as a $*$-algebra by the matrix elements of π;
(iv) $\mathscr{R}(G)$ is generated as an algebra by the matrix elements of π and by $(\det \circ \pi)^{-1}$.

13.8. Show that the following conditions are equivalent:

(i) G is topologically isomorphic to a Lie group;
(ii) G has a finite-dimensional faithful continuous representation;
(iii) $\mathscr{R}(G)$ is a finitely generated algebra.

(You may use the fact that a closed subgroup of a Lie group is a smooth submanifold.)

13.9. For each $n \in \mathbb{Z}_{\geqslant 0}$, let L_n denote the space of complex homogeneous polynomials of degree n in two variables. Define a representation π_n of $\mathrm{SU}(2)$ on L_n by $(\pi_n(x)f)(u) = f(ux)$ (here ux stands for the product of the row $u = (u_1, u_2) \in \mathbb{C}^2$ by the 2×2 matrix $x \in \mathrm{SU}(2)$).
(a) Prove that π_n is irreducible.
(b) Calculate the restriction of the character $\chi_n = \chi_{\pi_n}$ to the diagonal subgroup $\mathbb{T}$ of $\mathrm{SU}(2)$.
(c) Prove that the character of $\pi_m \otimes \pi_n$ (where $m \leqslant n$) is equal to $\sum_{k=0}^{m} \chi_{m+n-2k}$.
(d) Deduce that $\widehat{\mathrm{SU}(2)} = \{\pi_n : n \in \mathbb{Z}_{\geqslant 0}\}$.

Hints. **(a)** Consider the restriction of π_n to $\mathbb{T}$ and show that each invariant subspace $E \subset L_n$ is spanned by monomials. Then act by a nondiagonal matrix $x \in \mathrm{SU}(2)$ on a monomial $u_1^i u_2^j \in E$.
(c) A central function on $\mathrm{SU}(2)$ is completely determined by its restriction to $\mathbb{T}$.
(d) Use (c) and Exercise 13.6.

13.10. Suppose that the topology on G has a countable base. Prove that $\widehat{G}$ is at most countable.

Define the *Fourier cotransform* $\check{\mathscr{F}} \colon \bigoplus_{\sigma\in\widehat{G}} \operatorname{End}(E^\sigma) \to \mathscr{R}(G)$ by $T \in \operatorname{End}(E^\sigma) \mapsto \pi_T^\sigma \in \mathscr{R}(G)$. Since, for each $\sigma \in \widehat{G}$, $\check{\mathscr{F}}$ maps $\operatorname{End}(E^\sigma)$ isomorphically onto $\mathscr{R}_\sigma(G)$, and since $\mathscr{R}(G) = \bigoplus_\sigma \mathscr{R}_\sigma(G)$ (see the lectures), it follows that $\check{\mathscr{F}}$ is an isomorphism.

13.11. Construct a vector space isomorphism $(\bigoplus_\sigma \operatorname{End}(E^\sigma))^* \cong \prod_\sigma \operatorname{End}(E^\sigma)$ (where the star denotes the *algebraic* dual space).

Let $\mathscr{F} = (\check{\mathscr{F}})^* \colon \mathscr{R}(G)^* \to \prod_\sigma \operatorname{End}(E^\sigma)$ be the (algebraic) dual of $\check{\mathscr{F}}$. Morally, the elements of $\mathscr{R}(G)^*$ should be interpreted as "generalized distributions" on G, and $\mathscr{F}$ is the Fourier transform of generalized distributions[1].

13.12. Define $i \colon L^1(G) \to \mathscr{R}(G)^*$ by $i(f)(g) = \int_G fg\, d\mu$.
(a) Prove that i is injective.
(b) Show that the following diagram commutes:

$$\begin{array}{ccc} \mathscr{R}(G)^* & \xrightarrow{\ \mathscr{F}\ } & \prod_{\sigma\in\widehat{G}} \operatorname{End}(E^\sigma) \\ \uparrow{\scriptstyle i} & & \uparrow \\ L^1(G) & \xrightarrow{\ \mathscr{F}\ } & \left(\bigoplus_{\sigma\in\widehat{G}} \operatorname{End}(E^\sigma)\right)_0 \end{array}$$

This justifies the notation $\mathscr{F}$ for the Fourier transform on $\mathscr{R}(G)^*$ (compare with Exercise 4.11 (a)).

[1]If G is a compact Lie group, then the space $\mathscr{E}'(G)$ of distributions on G is defined to be the topological dual of $C^\infty(G)$. In this case, $\mathscr{R}(G)$ is a dense subspace of $C^\infty(G)$, so we have an embedding $\mathscr{E}'(G) \hookrightarrow \mathscr{R}(G)^*$.