11.1. Let G be a topological group, and let E be a Banach G-module. Recall that the standard G-module structure on E^* is given by $(xf)(v) = f(x^{-1}v)$ (where $x \in G$, $f \in E^*$, $v \in E$). Give an example showing that the action $G \times E^* \to E^*$ is not necessarily continuous.

11.2. (a) Let G be a topological group, and let H be a unitary irreducible G-module. Show that the inner product on H is a unique (up to a multiplicative constant) G-invariant continuous sesquilinear form on H.
(b) Deduce from (a) that, if two unitary irreducible G-modules H_1 and H_2 are topologically isomorphic (i.e., if there is a Banach space isomorphism $H_1 \to H_2$ which is a G-module isomorphism), then H_1 and H_2 are unitarily isomorphic.

11.3. Extend 11.2 (b) to arbitrary (i.e., not necessarily irreducible) unitary G-modules. (*Hint:* given a topological isomorphism $T\colon H_1 \to H_2$, use the polar decomposition $T = US$, where $S = (T^*T)^{1/2}$.)

11.4. Let G be a locally compact group. Prove that the left and right regular representations of G on $L^2(G)$ are faithful.

11.5. Let G be a group, and let E, F be finite-dimensional G-modules. Show that the canonical isomorphism $F \otimes E^* \cong \mathrm{Hom}(E, F)$ is a $G \times G$-module isomorphism.

11.6. Let G be a group, and let E be a finite-dimensional G-module. Show that E is irreducible iff E^* is irreducible.

11.7. Let G be a group, and let E, F be finite-dimensional irreducible G-modules. Prove that $E \otimes F$ is an irreducible $G \times G$-module.

11.8. Let G be a group, and let E be a unitary G-module. Show that the canonical isomorphism $\bar{E} \xrightarrow{\sim} E^*$, $\bar{v} \mapsto (u \mapsto \langle u \,|\, v\rangle)$, is a G-module isomorphism.

11.9. Let G be a topological group, and let G act on $C(G)$ via the left (or right) regular representation. Prove that
(a) the action of G on any finite-dimensional submodule of $C(G)$ is continuous;
(b) if G is compact, then the action of G on $C(G)$ is continuous (here $C(G)$ is equipped with the sup norm).

11.10. Let G be a topological group, and let (E, π) be a continuous finite-dimensional irreducible representation of G. Show that $\pi(G)$ spans $\mathrm{End}(E)$ as a vector space.

Given a topological group G, define $\Delta\colon C(G) \to C(G \times G)$ by $(\Delta f)(x, y) = f(xy)$. Identify $C(G) \otimes C(G)$ with a subspace of $C(G \times G)$ via the embedding $f \otimes g \mapsto ((x, y) \mapsto f(x)g(y))$. Recall that the algebra $\mathscr{R}(G)$ of *representative functions* on G is given by $\mathscr{R}(G) = \Delta^{-1}(C(G) \otimes C(G))$. Also recall (see the lectures) that a continuous function $f\colon G \to \mathbb{C}$ belongs to $\mathscr{R}(G)$ iff the left (or, equivalently, right) translates of f span a finite-dimensional subspace of $C(G)$. Also recall (again see the lectures) that $\mathscr{R}(G)$ consists of matrix elements of finite-dimensional continuous representations of G.

11.11. (a) Show that $\Delta(\mathscr{R}(G)) \subset \mathscr{R}(G) \otimes \mathscr{R}(G)$.
(b) If you know what a Hopf algebra is, show that $\mathscr{R}(G)$ is a Hopf algebra with comultiplication Δ, counit $\varepsilon\colon \mathscr{R}(G) \to \mathbb{C}$, $f \mapsto f(e)$, and antipode $S\colon \mathscr{R}(G) \to \mathscr{R}(G)$, $(Sf)(x) = f(x^{-1})$.

11.12. Let G be an infinite locally compact group. Show that $\mathscr{R}(G)$ is a proper subalgebra of $C(G)$.