9.1. Let π be a representation of a topological group G on a finite-dimensional vector space E. We endow E and $\mathrm{GL}(E)$ with the standard (normed) topologies. Prove that π is continuous (i.e., the map $G \times E \to E$, $(x, v) \mapsto \pi(x)v$, is continuous) if and only if π is a continuous map from G to $\mathrm{GL}(E)$.

9.2. Let π be a representation of a locally compact group G on a Banach space E. Let $\mathrm{GL}_{\mathrm{top}}(E)$ denote the group of all linear topological automorphisms of E. Prove that π is continuous (i.e., the map $G \times E \to E$, $(x, v) \mapsto \pi(x)v$, is continuous) if and only if π is a continuous map from G to $\mathrm{GL}_{\mathrm{top}}(E)$ equipped with the strong operator topology.

9.3. Let G be a nondiscrete locally compact group. Show that the left regular representation of G on $L^2(G)$ is a discontinuous map from G to $\mathrm{GL}_{\mathrm{top}}(G)$ for the operator norm topology on $\mathrm{GL}_{\mathrm{top}}(E)$.

9.4. Let G be a locally compact group, and let $1 \leqslant p < \infty$. Show that $C_c(G)$ is a dense subspace of $L^p(G)$. (This result was used at the lecture when we constructed the regular representation of G on $L^p(G)$.)

9.5. Define the left regular representation λ of a locally compact group G on $L^\infty(G)$ in exactly the same way as on $L^p(G)$ $(1 \leqslant p < \infty)$. Is λ necessarily continuous? Is the restriction of λ to $C_b(G)$ or to $C_0(G)$ continuous?

9.6. Let G be a locally compact group. Construct a unitary isomorphism between the left and right regular representations of G on $L^2(G)$.

9.7. (a) Let S be a locally compact Hausdorff topological space equipped with a continuous action $G \times S \to S$ of a locally compact group G. Suppose that μ is a G-invariant Radon measure on S (i.e., $\mu(xB) = \mu(B)$ for each Borel set $B \subset S$ and each $x \in G$). Prove that the formula $(\pi(x)f)(s) = f(x^{-1}s)$ $(x \in G,\ s \in S,\ f \in L^2(S, \mu))$ determines a continuous representation π of G on $L^2(S, \mu)$.
(b) Let $G = \mathrm{SU}(2)$ act on the sphere $S = S^3 \subset \mathbb{C}^2$ tautologically, and let π denote the respective representation of G on $L^2(S, \mu)$, where μ is the standard area measure on S. Construct a unitary isomorphism between π and the left regular representation of G.

9.8. Show that $\mathrm{SL}(2, \mathbb{R})$ has no finite-dimensional unitary representations except for the trivial one.

Hint. Conjugate the matrix $A(t) = \left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)$ by the matrix $\left(\begin{smallmatrix} n & 0 \\ 0 & 1/n \end{smallmatrix}\right)$ $(t \in \mathbb{R},\ n \in \mathbb{N})$, then apply a unitary finite-dimensional representation π, and look at the spectrum of $\pi(A(t))$.

9.9. Let G be the Heisenberg group (see Exercise 5.9). Define a representation π of G on $L^2(\mathbb{R})$ by

$$\left(\pi \left(\begin{smallmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{smallmatrix}\right) f\right)(x) = e^{2\pi i(bx+c)} f(x+a).$$

Show that π is unitary and irreducible. Is it algebraically irreducible?

Hint. If $H_0 \subset L^2(\mathbb{R})$ is a closed G-submodule, then H_0 is invariant under translations and under multiplication by unitary characters of $\mathbb{R}$. Deduce that H_0 is invariant under convolution with functions belonging to $L^1(\mathbb{R})$ and under multiplication by functions in $C_0(\mathbb{R})$.

9.10. Let H denote the space of functions $f \colon \mathscr{H} \to \mathbb{C}$ that are holomorphic on the upper half-plane $\mathscr{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ and have the property that $|f|^2$ is Lebesgue integrable on $\mathscr{H}$. We endow H with the inner product inherited from $L^2(\mathscr{H})$. Define a representation π of $\mathrm{SL}(2, \mathbb{R})$ on H by

$$\left(\pi \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) f\right)(z) = (-bz + d)^{-2} f\left(\frac{az - c}{-bz + d}\right).$$

Show that π is unitary and irreducible. Is it algebraically irreducible?

Hint. Show that each closed $\mathrm{SL}(2,\mathbb{R})$-submodule $H_0 \subset H$ contains a function f such that $f(i) \neq 0$. Then calculate the integral

$$\int_0^{2\pi} e^{-2i\varphi}\, \pi\left(\begin{smallmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{smallmatrix}\right) f\, d\varphi$$

by using residues.

9.11. Define $f\colon [0,1] \to c_0$ by

$$f(t) = (\chi_{(0,1]}(t), 2\chi_{(0,1/2]}(t), \ldots, n\chi_{(0,1/n]}(t), \ldots) \qquad (t \in [0,1]).$$

Show that f is Dunford integrable (w.r.t. the Lebesgue measure on $[0,1]$), but is not Pettis integrable.

9.12. Let (X,μ) be a measure space, and let E, F be Banach spaces.
(a) Suppose that $f\colon X \to E$ is Dunford integrable and that $\|f\|\colon x \mapsto \|f(x)\|$ is integrable. Show that $\|\int f\, d\mu\| \leqslant \int \|f\|\, d\mu$.
(b) Suppose that $f\colon X \to E$ is Pettis integrable. Show that for each bounded linear map $T\colon E \to F$ the function $T \circ f$ is Pettis integrable, and that $T(\int f\, d\mu) = \int (T \circ f)\, d\mu$.