

5.1. Let $(X_i)_{i \in I}$ be a family of topological spaces. Show that $\prod_{i \in I} X_i$ is locally compact if and only if each X_i is locally compact and only finitely many of the X_i 's are noncompact.

5.2. Show that the counting measure on a nondiscrete Hausdorff topological space is not a Radon measure.

5.3. Let X be a locally compact, σ -compact Hausdorff topological space. Show that each outer Radon measure on X is inner regular on all Borel sets.

5.4. Let X be a locally compact, second countable Hausdorff topological space. Show that each Borel measure on X that is finite on compact sets is a Radon measure.

5.5. Let X be a locally compact topological space, and let $C_c(X)$ denote the space of all continuous compactly supported functions on X . Given a compact set $K \subset X$, let $C_K(X) = \{f \in C_c(X) : \text{supp } f \subset K\}$. We endow $C_K(X)$ with the topology generated by the sup-norm $\|f\| = \sup_{x \in K} |f(x)|$. Show that every positive linear functional $I: C_c(X) \rightarrow \mathbb{C}$ is continuous on $C_K(X)$, for each compact set $K \subset X$. (Equivalently, this means that I is continuous w.r.t. the inductive limit topology on $C_c(X) = \varinjlim_K C_K(X)$.) Of course, you are not allowed to use the Riesz-Markov-Kakutani theorem, otherwise the exercise would be trivial. . .

5.6. Give an example of a locally compact group G and a left uniformly continuous function $f: G \rightarrow \mathbb{C}$ that is not right uniformly continuous.

5.7. Find explicitly the left and the right Haar measures on $\text{GL}(n, \mathbb{R})$. (You can find an answer in many books, and you can easily check that it works, but I recommend you to deduce the formula for the Haar measure by using the method that was discussed at the lecture in the context of arbitrary Lie groups.)

5.8 (*the "ax + b" group*). Let G be the group of all matrices of the form $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$, where $a \in \mathbb{R}^\times$ and $b \in \mathbb{R}$ (this group is isomorphic to the group of all affine transformations $x \mapsto ax + b$ of \mathbb{R}). Find explicitly the left and the right Haar measures on G . (You can find an answer in many books, and you can easily check that it works, but I recommend you to deduce the formula for the Haar measure by using the method that was discussed at the lecture in the context of arbitrary Lie groups.)

5.9 (*the Heisenberg group*). Let G be the group of all matrices of the form $\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$, where $a, b, c \in \mathbb{R}$. Find explicitly the left and the right Haar measures on G . (You can find an answer in many books, and you can easily check that it works, but I recommend you to deduce the formula for the Haar measure by using the method that was discussed at the lecture in the context of arbitrary Lie groups.)

5.10. Let G be a locally compact group, and let $\chi: G \rightarrow \mathbb{R}_{>0}$ be a continuous homomorphism. Show that there exists a unique (up to a positive constant) positive Radon measure on G such that for each Borel set $B \subset G$ we have $\mu(xB) = \chi(x)\mu(B)$. (*Hint*: express μ in terms of a Haar measure on G .)

5.11. Let $p \in \mathbb{N}$ be a prime number. Show that the following definitions of the field \mathbb{Q}_p of p -adic numbers and of the ring \mathbb{Z}_p of p -adic integers are equivalent:

- (i) $\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^n\mathbb{Z}$, \mathbb{Q}_p is the field of fractions of \mathbb{Z}_p .
- (ii) \mathbb{Q}_p is the completion of \mathbb{Q} w.r.t. the p -adic norm $|\cdot|_p$ given by $|x|_p = p^{-r}$, where $x = p^r a/b \in \mathbb{Q} \setminus \{0\}$, $a \in \mathbb{Z}$, $b \in \mathbb{N}$, $p \nmid a$, $p \nmid b$; $\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$.

- (iii) \mathbb{Q}_p consists of all formal expressions of the form $x = \sum_{k=n}^{\infty} a_k p^k$, where $n \in \mathbb{Z}$ and $a_k \in \{0, 1, \dots, p-1\}$; $\mathbb{Z}_p = \{x \in \mathbb{Q}_p : a_k = 0 \forall k < 0\}$. (Define algebraic operations on such formal expressions! What is $|x|_p$ if x has the above form?)

Show that the projective limit topology on \mathbb{Z}_p defined in (i) agrees with the norm topology defined in (ii). Prove that \mathbb{Z}_p is compact and that \mathbb{Q}_p is locally compact.

5.12. Let μ denote the Haar measure on \mathbb{Q}_p normalized in such a way that $\mu(\mathbb{Z}_p) = 1$. Show that

- (a) $\mu(\mathbb{B}_{p^k}(x)) = p^{-k}$, where $\mathbb{B}_{p^k}(x)$ is the closed ball of radius p^k ($k \in \mathbb{Z}$) centered at $x \in \mathbb{Q}_p$.
 (b) For each Borel set $B \subset \mathbb{Q}_p$ we have

$$\mu(B) = \inf \left\{ \sum_{i=1}^{\infty} p^{-k_i} : B \subset \bigcup_{i=1}^{\infty} \mathbb{B}_{p^{k_i}}(x_i) \right\}.$$

5.13. Let $G = (\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$ be the product of countably many copies of $\mathbb{Z}/2\mathbb{Z}$, and let μ denote the normalized Haar measure on G .

- (a) Calculate $\mu(U)$ for a basic open set $U = \prod_{i=1}^{\infty} U_i$, where $U_i = \mathbb{Z}/2\mathbb{Z}$ for all but finitely many i .
 (b) Define $f: G \rightarrow [0, 1]$ by $f(a_1, a_2, \dots) = \sum_i a_i 2^{-i}$. Show that f is onto and that $f^{-1}(x)$ is one point unless x is a dyadic rational, in which case $f^{-1}(x)$ consists of two points. Prove that the image of μ under f is the Lebesgue measure on $[0, 1]$.
 (c) Define $h: G \rightarrow [0, 1]$ by $h(a_1, a_2, \dots) = \sum_i 2a_i 3^{-i}$. Show that h is a homeomorphism of G onto the Cantor set. Prove that the image of μ under h is the Lebesgue-Stieltjes measure on $[0, 1]$ associated to the Cantor function.