

**3.1.** Prove that the homomorphisms  $\mathbb{T} \rightarrow \widehat{\mathbb{Z}}$ ,  $\mathbb{Z} \rightarrow \widehat{\mathbb{T}}$ , and  $\mathbb{R} \rightarrow \widehat{\mathbb{R}}$  that were constructed at the lecture are indeed topological isomorphisms.

**3.2.** Prove that the characters of  $\mathbb{T}$  form an orthonormal set in  $L^2(\mathbb{T})$ .

**3.3.** Prove that the Fourier transform  $\mathcal{F}_{\mathbb{T}}: L^1(\mathbb{T}) \rightarrow c_0(\mathbb{Z})$  is not surjective. (*Hint:* the dual spaces of  $L^1(\mathbb{T})$  and  $c_0(\mathbb{Z})$  are not isomorphic.)

For each  $k \in \mathbb{Z}$  define  $\chi_k: \mathbb{T} \rightarrow \mathbb{T}$  by  $\chi_k(z) = z^{-k}$  (recall that the functions  $\chi_k$  are precisely the characters of  $\mathbb{T}$ ). The function  $D_n = \sum_{k=-n}^n \chi_k$  on  $\mathbb{T}$  is called the *Dirichlet kernel*. For an arbitrary  $f \in L^1(\mathbb{T})$  set  $S_n f = \sum_{k=-n}^n \hat{f}(k) \chi_{-k}$ .

**3.4. (a)** Prove that for each  $f \in L^1(\mathbb{T})$  we have  $S_n f = f * D_n$ . (*Hint:* compare the Fourier transforms of these functions).

**(b)** Let  $g \in L^1(\mathbb{T})$ . Define a functional  $\varphi_g: C(\mathbb{T}) \rightarrow \mathbb{C}$  by the formula  $\varphi_g(f) = \int_{\mathbb{T}} f(z)g(z) d\mu(z)$ . Prove that  $\|\varphi_g\| = \|g\|_1$ .

**(c)** Prove that

$$D_n(e^{it}) = 1 + 2 \cos t + 2 \cos 2t + \cdots + 2 \cos nt = \frac{\sin(n + \frac{1}{2})t}{\sin \frac{t}{2}}.$$

Deduce that  $\|D_n\|_1 \rightarrow \infty$  as  $n \rightarrow \infty$ .

**(d)** Prove that there exists a function  $f \in C(\mathbb{T})$  whose Fourier series diverges at  $z = 1$ . It follows that  $\mathcal{F}_{\mathbb{Z}}: \ell^1(\mathbb{Z}) \rightarrow C(\mathbb{T})$  is not surjective. (*Hint:* It follows from (a) that  $(S_n f)(1) = \varphi_{D_n}(f)$ . Then use (b), (c) and the Banach-Steinhaus theorem.)

**3.5.** Compute  $\|\hat{D}_n\|_{\infty}$ . Combine this with part (c) of the previous exercise to find another proof of the fact that the Fourier transform  $\mathcal{F}_{\mathbb{T}}: L^1(\mathbb{T}) \rightarrow c_0(\mathbb{Z})$  is not surjective.

**3.6\*.** Prove that the series

$$\sum_{n=2}^{\infty} \frac{z^n + z^{-n}}{n \ln n}$$

converges uniformly on  $\mathbb{T}$  to a continuous function whose Fourier series is not absolutely convergent. This gives yet another proof of the fact that the Fourier transform  $\mathcal{F}_{\mathbb{Z}}: \ell^1(\mathbb{Z}) \rightarrow C(\mathbb{T})$  is not surjective.

**3.7.** Fix  $a \in \mathbb{T}$  and for any  $f \in L^2(\mathbb{T})$  and for any  $n \in \mathbb{N}$  define

$$f_n(z) = \frac{1}{n} \sum_{k=0}^{n-1} f(a^k z).$$

Prove that the sequence  $(f_n)$  converges in  $L^2(\mathbb{T})$  to some  $f$ . Find  $f$ .

**3.8.** The *convolution* of  $f, g \in \ell^1(\mathbb{Z})$  is the function  $f * g: \mathbb{Z} \rightarrow \mathbb{C}$  defined by the formula

$$(f * g)(n) = \sum_{k \in \mathbb{Z}} f(k)g(n - k) \quad (n \in \mathbb{Z}). \quad (1)$$

**(a)** Prove that the series (1) converges absolutely, that  $f * g \in \ell^1(\mathbb{Z})$ , that  $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$ , and that  $(\ell^1(\mathbb{Z}), *)$  is a commutative unital algebra containing the group algebra  $\mathbb{C}\mathbb{Z}$  as a subalgebra.

**(b)** Prove that for any  $f, g \in \ell^1(\mathbb{Z})$  we have  $(f * g)^{\hat{}} = \hat{f} \hat{g}$  (i.e., the Fourier transform  $\mathcal{F}: \ell^1(\mathbb{Z}) \rightarrow C(\mathbb{T})$  is an algebra homomorphism).

**3.9.** The *convolution* of  $f, g \in L^1(\mathbb{T})$  is the function  $f * g: \mathbb{T} \rightarrow \mathbb{C}$  defined by the formula

$$(f * g)(z) = \int_{\mathbb{T}} f(\zeta)g(\zeta^{-1}z) d\mu(\zeta) \quad (z \in \mathbb{T}), \quad (2)$$

where  $\mu$  is the standard measure on  $\mathbb{T}$  (arch length/ $2\pi$ ).

**(a)** Prove that the integral (2) exists for almost all  $z \in \mathbb{T}$ , that  $f * g \in L^1(\mathbb{T})$ , that  $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$ , and that  $(L^1(\mathbb{T}), *)$  is a commutative non-unital algebra.

**(b)** Prove that for any  $f, g \in L^1(\mathbb{T})$  we have  $(f * g)^\wedge = \hat{f}\hat{g}$  (i.e., the Fourier transform  $\mathcal{F}: L^1(\mathbb{T}) \rightarrow c_0(\mathbb{Z})$  is an algebra homomorphism).