Convention. All vector spaces are over $\mathbb{C}$.

1.1. Prove that every character of a finite group is unitary.

1.2. Prove that a finite group has an injective character iff it is cyclic.

1.3. Describe all characters of **(a)** S_n; **(b)** D_n; **(c)** Q_8.

1.4. Prove that, for a finite group, the intersection of the kernels of all the characters is the commutator subgroup (that is, the subgroup generated by all possible commutators $xyx^{-1}y^{-1}$).

1.5. Let G be a finite group, and let $\mathrm{Fun}(G) = \mathbb{C}^G$ be the space of all functions on G. Recall that the convolution on $\mathrm{Fun}(G)$ is a bilinear map uniquely determined by $\delta_x * \delta_y = \delta_{xy}$ ($x, y \in G$), where δ_x is the function equal to 1 at $x \in G$ and 0 elsewhere. Prove that for all $f, g \in \mathrm{Fun}(G)$ we have

$$(f * g)(x) = \sum_{y \in G} f(y) g(y^{-1}x) \qquad (x \in G).$$

1.6. Let G be a finite abelian group of order n, and let $\pi \colon G \to \mathrm{GL}(V)$ be a representation on a finite-dimensional vector space V. For every $\chi \in \widehat{G}$ define $V_\chi = \{v \in V : \pi(x)v = \chi(x)v \ \forall x \in G\}$. Define an operator P_χ on V by

$$P_\chi = \frac{1}{n} \sum_{x \in G} \overline{\chi(x)} \pi(x).$$

(a) Prove that $P_\chi P_\tau = \delta_{\chi\tau} P_\chi$, $\sum_{\chi \in \widehat{G}} P_\chi = \mathbf{1}_V$, and $\operatorname{Im} P_\chi = V_\chi$. Deduce that $V = \bigoplus_{\chi \in \widehat{G}} V_\chi$ and that P_χ is a projection onto V_χ along $\bigoplus_{\tau \neq \chi} V_\tau$.
(b) Find V_χ in the case where π is the regular representation of G on $\mathrm{Fun}(G)$ given by $(\pi(x)f)(y) = f(yx)$.

1.7. Let G be a finite abelian group. Prove that the following properties of a linear operator $T \colon \mathrm{Fun}(G) \to \mathrm{Fun}(G)$ are equivalent:

(i) T is shift invariant (i.e., $T\pi(x) = \pi(x)T$ for all $x \in G$, where π is the regular representation from the previous exercise);
(ii) there exists a function $f \in \mathrm{Fun}(G)$ such that $Th = f * h$ for all $h \in \mathrm{Fun}(G)$;
(iii) all characters of G are eigenvectors for T.

1.8. Let G be a finite abelian group, and let H be a subgroup of G.
(a) Construct an isomorphism $(G/H)\widehat{\ } \cong H^\perp$, where $H^\perp = \{\chi \in \widehat{G} : \chi|_H = 0\}$ is the *annihilator* of H in $\widehat{G}$.
(b) (*the Poisson summation formula*). Define $a \colon \mathrm{Fun}(G) \to \mathrm{Fun}(G/H)$ by $(af)(xH) = \sum_{y \in H} f(xy)$. Show that the diagram

$$\begin{array}{ccc} \mathrm{Fun}(G) & \xrightarrow{\mathscr{F}_G} & \mathrm{Fun}(G) \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle \text{restr.}} \\ \mathrm{Fun}(G/H) & \xrightarrow[\mathscr{F}_H]{} & \mathrm{Fun}(H^\perp) \end{array}$$

commutes. Deduce that for each $f \in \mathrm{Fun}(G)$ we have

$$\sum_{y \in H} f(xy) = \frac{1}{(G:H)} \sum_{\chi \in H^\perp} \hat{f}(\chi) \overline{\chi(x)} \qquad (x \in G).$$