Convention. All vector spaces are over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

Embeddings and quotients

Let X and Y be normed spaces. Recall that a linear operator $T\colon X \to Y$ is a *coisometry* if it takes the open unit ball of X onto the open unit ball of Y.

1.1. Let X be a normed space, and let $f\colon X \to \mathbb{K}$ be a linear functional.
(a) Show that f is open (unless $f = 0$).
(b) Show that f is a coisometry iff $\|f\| = 1$.
Warning: do not forget about the case where $\mathbb{K} = \mathbb{C}$.

1.2. Let X and Y be normed spaces, and let $T\colon X \to Y$ be a linear operator.
(a) Prove that if T maps the closed unit ball of X onto the closed unit ball of Y, then T is a coisometry.
(b) Is the converse true?
(c) Show that T is an injective coisometry iff T is an isometric isomorphism.

1.3. Let $\alpha \in \ell^\infty$, and let X denote either ℓ^p or c_0. Let M_α be the *diagonal operator* on X defined by $M_\alpha(x) = (\alpha_i x_i)_{i\in\mathbb{N}}$. Find a condition on α that is necessary and sufficient for M_α to be
(a) topologically injective; **(b)** open; **(c)** an isometry; **(d)** a coisometry.

1.4. Let (Ω, μ) be a σ-finite measure space, and let f be a bounded measurable function on Ω. Answer questions **(a)** – **(d)** of the previous exercise for the *multiplication operator* M_f on $L^p(\Omega, \mu)$, $M_f(g) = fg$.

1.5. Let X be a normed space, and let $X_0 \subset X$ be a vector subspace. Prove that
(a) the quotient seminorm on X/X_0 is indeed a seminorm;
(b) the topology on X/X_0 determined by the quotient seminorm is the quotient topology (i.e., a subset $U \subset X/X_0$ is open iff $Q^{-1}(U)$ is open in X, where $Q\colon X \to X/X_0$ is the quotient map).

1.6. Construct **(a)** a topological isomorphism between c_0 and a quotient of $C[0,1]$; **(b)** an isometric isomorphism between ℓ^1 and a quotient of $L^1[0,1]$.

Duality for normed spaces

Let X, Y be normed spaces, and let $T\colon X \to Y$ be a bounded linear operator. Recall (see the lectures) that the *dual* of T is the operator $T^*\colon Y^* \to X^*$ given by $T^* f = f \circ T$ (where $f \in Y^*$).

In the following exercise you are supposed to describe explicitly the duals of some concrete operators. The phrase "describe explicitly" means the following. Given an operator $T\colon X \to X$, where X is a Banach space, find another "classical" Banach space Y together with an isometric isomorphism $u\colon Y \xrightarrow{\sim} X^*$ (hint: Y and u were discussed at the lectures in the previous term). Then find (define by an explicit formula) an operator $S\colon Y \to Y$ such that the following diagram commutes:

$$\begin{array}{ccc} Y & \xrightarrow{S} & Y \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle u} \\ X^* & \xrightarrow{T^*} & X^* \end{array}$$

1.7. Describe explicitly the duals of the following operators:
(a) the diagonal operator M_α on ℓ^p (where $1 \leqslant p < \infty$) or on c_0 (see Exercise 1.3);
(b) the *right shift* operator T_r and the *left shift* operator T_ℓ on ℓ^p ($1 \leqslant p < \infty$) or on c_0 given by

$$T_r(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots);$$
$$T_\ell(x_1, x_2, \ldots) = (x_2, x_3, \ldots)$$

(c) the *Volterra integral operator* V on $L^p[0,1]$ (where $1 \leqslant p < \infty$) given by

$$(Vf)(x) = \int_0^x f(t)\,dt \qquad (x \in [0,1]);$$

(d) the *Hilbert-Schmidt integral operator* T_K on $L^2(X,\mu)$ given by

$$(T_K f)(x) = \int_X K(x,y) f(y)\,d\mu(y) \qquad (x \in X)$$

where (X,μ) is a measure space and $K \in L^2(X \times X, \mu \times \mu)$.

1.8. Let X be a normed space, and let $i_X \colon X \to X^{**}$ be the canonical embedding. Prove that for each operator $T \in \mathscr{B}(X,Y)$ the following diagram commutes.

$$\begin{array}{ccc} X^{**} & \xrightarrow{T^{**}} & Y^{**} \\ {\scriptstyle i_X}\uparrow & & \uparrow{\scriptstyle i_Y} \\ X & \xrightarrow{T} & Y \end{array}$$

1.9. Let X be a normed space, and let $i_X \colon X \to X^{**}$ be the canonical embedding.
(a) Find a relation between the operators $i_{X^*} \colon X^* \to X^{***}$ and $i_X^* \colon X^{***} \to X^*$.
(b) Prove that a Banach space X is reflexive $\iff$ X^* is reflexive.
(c) Deduce that ℓ^1, ℓ^∞, $L^\infty[a,b]$ are not reflexive.

1.10. Let X and Y be Banach spaces, and let $S \in \mathscr{B}(Y^*, X^*)$. Do we always have $S = T^*$ for some $T \in \mathscr{B}(X,Y)$?

Let X be a normed space. Recall (see the lectures) that the *annihilator* of a subset $M \subset X$ and the *preannihilator* of a subset $N \subset X^*$ are given by

$$M^\perp = \{f \in X^* : f(x) = 0 \;\forall\, x \in M\}, \quad {}^\perp N = \{x \in X : f(x) = 0 \;\forall\, f \in N\}.$$

The *double annihilator theorem* (see the lectures) asserts that for every $M \subset X$ we have ${}^\perp(M^\perp) = \overline{\operatorname{span}}(M)$ (the closure of the linear span). The following two exercises show that the "dual" formula $({}^\perp N)^\perp = \overline{\operatorname{span}}(N)$ fails in general (unless X is reflexive).

1.11. Identify $(\ell^1)^*$ with ℓ^∞, and consider c_0 as a subspace of $(\ell^1)^*$. Find ${}^\perp c_0$ and $({}^\perp c_0)^\perp$.

1.12. Let X be a nonreflexive Banach space. Prove that there exists a closed vector subspace $N \subset X^*$ such that $N \neq ({}^\perp N)^\perp$.

Recall (see the lectures) that for every bounded linear operator $T \colon X \to Y$ between normed spaces we have $\overline{\operatorname{Im} T} = {}^\perp(\operatorname{Ker} T^*)$. In particular, $\operatorname{Im} T$ is dense in Y iff T^* is injective. The following exercise shows that the "dual" statements fail in general (unless X is reflexive).

1.13. Give an example of an injective operator $T \in \mathscr{B}(X,Y)$ between Banach spaces X and Y such that $\operatorname{Im} T^*$ is not dense in X^*. (*Hint:* X must be nonreflexive, see above.) As a corollary, the equality $\overline{\operatorname{Im}(T^*)} = (\operatorname{Ker} T)^\perp$ fails.

1.14. Let X be a Banach space, and let $X_0 \subset X$ be a closed vector subspace. Prove that X is reflexive if and only if X_0 and X/X_0 are reflexive. (*Hint:* Johnson's lemma.)