Functional Analysis 2

Exam 26.05.2023 and 02.06.2023

1. **Topological vector spaces**
   - Topological vector spaces. The topology generated by a family of seminorms. Locally convex spaces and their "polynormability". Examples of locally convex spaces.
   - A characterization of convergence in a locally convex space in terms of seminorms.
   - A criterion for a locally convex space to be Hausdorff.
   - A continuity criterion for a seminorm on a locally convex space.
   - A continuity criterion for a linear operator between locally convex spaces.
   - The domination relation for families of seminorms. Equivalent families of seminorms. Examples.
   - Quotients of locally convex spaces.
   - Linear functionals on locally convex spaces (extension from subspaces, separation of points, separation of points and subspaces).
2. **Dual pairs and weak topologies**
   - Dual pairs of vector spaces. The weak topology of a dual pair. Special cases and basic properties of the weak topology.
   - A characterization of linear functionals that are continuous for the weak topology.
   - A characterization of reflexive Banach spaces in terms of topologies on the dual.
   - Dual operators between dual pairs of vector spaces. An existence criterion for the dual operator in terms of weak continuity.
   - Relations between the continuity, the weak continuity, and the existence of the dual for an operator between locally convex spaces (in particular, between normed spaces).
   - Annihilators. Basic properties of annihilators. The double annihilator theorem (for dual pairs and for locally convex spaces). Corollaries: the weak closure of a vector subspace; a density criterion for a vector subspace; relations between kernels and images of a linear operator and of the dual operator.
   - Equicontinuous families of linear operators. The Banach–Alaoglu–Bourbaki theorem.
3. **Commutative Banach algebras**
   - The maximal spectrum and the character space of a commutative unital algebra.
   - The closedness of maximal ideals of a commutative unital Banach algebra.
   - The 1-1 correspondence between the character space and the maximal spectrum of a commutative unital Banach algebra.
   - An invertibility criterion in terms of characters.
   - The Gelfand topology on the maximal spectrum.
   - The compactness of the maximal spectrum (unital case).
   - The Gelfand transform of a commutative unital Banach algebra.
   - Properties of the Gelfand transform.
   - The maximal spectrum and the Gelfand transform for subalgebras of $C(X)$.

4. **C^*-algebras and continuous functional calculus**

- The adjoint of an operator on a Hilbert space. Basic properties of the operation $T \mapsto T^*$.
- Banach $*$-algebras. C^*-algebras. Examples.
- Selfadjoint elements, unitary elements, normal elements.
- The main property of the spectral radius of a normal element. Corollaries: the uniqueness of the C^*-norm on a $*$-algebra; the automatic continuity of $*$-homomorphisms.
- The main property of the spectrum of a selfadjoint element in a C^*-algebra. Corollaries: characters of a C^*-algebra are $*$-characters; the spectral invariance of C^*-subalgebras.
- The 1st (commutative) Gelfand-Naimark theorem (unital case).
- The continuous functional calculus in C^*-algebras (definition, uniqueness, a necessary condition for the existence, the existence theorem).
- The spectral mapping theorem and the superposition property for the functional calculus.

5. **Spectral Theorem: functional models**

- Integration with respect to a complex measure. The Riesz-Markov-Kakutani theorem.
- $*$-representations and $*$-modules.
- A correspondence between normal operators and $*$-modules over $C(K)$ ($K \subset \mathbb{C}$). Relations between $*$-module morphisms and intertwining operators.
- Cyclic $*$-modules. Examples. The functional model for a cyclic $*$-module over $C(X)$.
- $*$-cyclic operators and their relation to cyclic $*$-modules.
- The functional model for a $*$-cyclic normal operator.
- Hilbert direct sums of Hilbert spaces and of $*$-modules.
- Decomposing $*$-modules into Hilbert direct sums of cyclic submodules.
- The functional model for a $*$-module over $C(X)$.
- The functional model for a normal operator (Spectral Theorem).