Functional Analysis 2

Midterm exam 24.03.2023

1. **Embeddings and quotients of normed spaces**
   - Topologically injective operators, isometries, topological and isometric isomorphisms. Topologically injective = bounded below.
   - Open operators and coisometries. Characterizations of open operators.
   - Quotients of normed spaces.
   - The universal property of quotients. Corollaries (in particular, conditions under which $X/\operatorname{Ker} T \cong \operatorname{Im} T$ topologically or isometrically).
   - The completeness of quotients.
2. **Duality for operators on Banach spaces**
   - Annihilators and preannihilators, their basic properties.
   - The double annihilator theorem. The density criterion for a vector subspace of a normed space.
   - The duals of subspaces and quotients.
   - Relations between the kernels and images of operators and of their duals.
   - A duality between injective operators and operators with dense image.
   - A duality between topologically injective and surjective operators.
   - A duality between isometries and coisometries.
   - The Closed Image Theorem.
   - Johnson's lemma on exact sequences of Banach spaces.
3. **Fredholm operators I: basic notions**
   - Fredholm operators. Fredholm index. Basic examples.
   - The additivity of the index.
   - Kato's lemma on the image of a Fredholm operator.
   - The fredholmness and the index of the dual operator.
4. **The Riesz–Schauder theory**
   - Schauder's theorem on the dual of a compact operator.
   - The ascent and the descent of a linear operator. Properties of linear operators with finite ascent and descent. The algebraic Riesz decomposition.
   - Riesz's theorem on operators "$\mathbf{1}$ + compact".
   - The Fredholm alternative. Abstract Fredholm theorems in Schauder's form.

5. **Spectral theory in Banach algebras**

    - The spectrum of an algebra element. Examples: the spectra of elements of $\mathbb{C}^X$, $\ell^\infty(X)$, $L^\infty(X, \mu)$.
    - The behavior of the spectrum under homomorphisms. Spectrally invariant subalgebras. Examples.
    - The polynomial spectral mapping theorem. The spectrum of the inverse element.
    - Banach algebras. Examples.
    - Properties of the group of invertibles in a Banach algebra. The automatic continuity of characters (i.e., of $\mathbb{C}$-valued homomorphisms).
    - Gelfand's theorem on the properties of the spectrum of a Banach algebra element.
    - The Gelfand–Mazur theorem.
    - The spectral radius of a Banach algebra element. Examples. The Beurling–Gelfand formula. Corollaries: a characterization of quasinilpotents, the spectral radius w.r.t. a closed subalgebra.

6. **Fredholm operators II: an interplay between Fredholm and compact operators**

    - Topological direct sums and complemented subspaces of normed spaces.
    - Characterizations of complemented subspaces in terms of projections and in terms of quotients.
    - Examples and counterexamples of complemented subspaces.
    - The Nikolskii–Atkinson criterion for Fredholm operators.
    - The Calkin algebra. The essential spectrum of a linear operator. The interpretation of the essential spectrum in terms of the Calkin algebra.
    - The stability of the index under "small" perturbations.
    - The stability of the index under compact perturbations.
    - Nikolskii's characterization of Fredholm operators of index zero.
    - Toeplitz operators on the Hardy space. The homomorphism $C(\mathbb{T}) \to \mathcal{Q}(H^2)$.
    - The index formula for a Toeplitz operator with continuous symbol.