Compact operators

Exercises marked by "**-B**" are optional. If you solve such exercises, you will earn bonus points.

1.1. Let X be a normed space, let $f \in X^* \setminus \{0\}$, and let $X_0 = \operatorname{Ker} f$. Show that there exists a 0-perpendicular to X_0 in X (that is, a vector $x \in X$ such that $\|x\| = \operatorname{dist}(x, X_0) = 1$) iff f is norm-attaining (which means that there exists $x \in X$, $\|x\| = 1$, such that $|f(x)| = \|f\|$). Give an example showing that this is not always the case.

1.2. (a) Prove that a subset $S \subset c_0$ is relatively compact iff there exists $y \in c_0$ such that $|x_n| \leqslant |y_n|$ for all $x \in S$ and all $n \in \mathbb{N}$. **(b)** Does a similar result hold for ℓ^p?

1.3. Are the left and right shift operators on ℓ^p and on c_0 compact?

1.4. Can the image of a compact operator between Banach spaces contain an infinite-dimensional closed vector subspace?

1.5. Prove that the inclusion $C^1[a,b] \to C[a,b]$ is a compact operator.

1.6. (a) Let $f \in C[a,b]$, and let M_f denote the respective multiplication operator on $C[a,b]$. Find a condition on f that is necessary and sufficient for M_f to be compact.
(b) Let $I \subset \mathbb{R}$ be an interval (not necessarily open or closed, not necessarily bounded), let $f\colon I \to \mathbb{C}$ be an essentially bounded measurable function, and let M_f denote the respective multiplication operator on $L^p(I)$ ($1 \leqslant p \leqslant \infty$). Find a condition on f that is necessary and sufficient for M_f to be compact.

1.7. Given an integrable function f on $[0,1]$, define a function Tf on $[0,1]$ by

$$(Tf)(x) = \int_0^x f(t)\,dt.$$

Is T a compact operator **(a)** from $C[0,1]$ to $C[0,1]$? **(b)** from $L^p[0,1]$ to $C[0,1]$ (where $1 < p \leqslant \infty$)? **(c)** from $L^p[0,1]$ to $L^p[0,1]$ (where $1 < p \leqslant \infty$)? **(d)** from $L^1[0,1]$ to $C[0,1]$? **(e)** from $L^1[0,1]$ to $L^1[0,1]$?

1.8. Let $I = [a,b]$, and let $K \in C(I \times I)$. Prove that the integral operator $T\colon C(I) \to C(I)$,

$$(Tf)(x) = \int_a^b K(x,y) f(y)\,dy,$$

is compact.

1.9. Let (X,μ) be a measure space, and let $K \in L^2(X \times X, \mu \times \mu)$. Prove that the *Hilbert–Schmidt integral operator* $T_K\colon L^2(X,\mu) \to L^2(X,\mu)$,

$$(T_K f)(x) = \int_X K(x,y) f(y)\,d\mu(y),$$

is compact.

Hint: show that functions of the form $K(x,y) = f(x)g(y)$, where $f, g \in L^2(X,\mu)$, span a dense subspace of $L^2(X \times X, \mu \times \mu)$, and use the fact that $\|T_K\| \leqslant \|K\|_2$ (see Exercise 2.8).

1.10. (a) Let X be a compact metrizable topological space, let $K \in C(X \times X)$, and let μ be a finite Borel measure on X. Show that the image of the Hilbert–Schmidt integral operator $T_K\colon L^2(X,\mu) \to L^2(X,\mu)$ is contained in $C(X)$, and that T_K is a compact operator from $L^2(X,\mu)$ to $C(X)$.
(b)-B Extend (a) to an arbitrary (not necessarily metrizable) compact topological space X.