Functional Analysis 2 (Operator Theory)

Exam 20.05.2022

1. **Dual pairs and weak topologies**
    - Dual pairs of vector spaces. The weak topology of a dual pair. Special cases and basic properties of the weak topology.
    - A characterization of linear functionals that are continuous for the weak topology.
    - A characterization of reflexive Banach spaces in terms of topologies on the dual.
    - Dual operators between dual pairs of vector spaces. An existence criterion for the dual operator in terms of weak continuity.
    - Relations between the continuity, the weak continuity, and the existence of the dual for an operator between locally convex spaces (in particular, between normed spaces).
    - Annihilators. Basic properties of annihilators. The double annihilator theorem (for dual pairs and for locally convex spaces). Corollaries: the weak closure of a vector subspace; a density criterion for a vector subspace; relations between kernels and images of a linear operator and of the dual operator.
    - Equicontinuous families of linear operators. The Banach–Alaoglu–Bourbaki theorem.
    - Polars and their basic properties. The bipolar theorem. Corollaries: Goldstine's theorem, a reflexivity criterion in terms of weak compactness.
2. **Commutative Banach algebras**
    - The maximal spectrum and the character space of a commutative unital algebra.
    - The closedness of maximal ideals of a commutative unital Banach algebra.
    - The 1-1 correspondence between the character space and the maximal spectrum of a commutative unital Banach algebra.
    - An invertibility criterion in terms of characters.
    - The Gelfand topology on the maximal spectrum.
    - The compactness of the maximal spectrum (unital case).
    - The Gelfand transform of a commutative unital Banach algebra.
    - Properties of the Gelfand transform.
    - The maximal spectrum and the Gelfand transform for subalgebras of $C(X)$.
3. **C^*-algebras and continuous functional calculus**
    - Banach $*$-algebras. C^*-algebras. Examples.
    - Selfadjoint elements, unitary elements, normal elements.
    - The main property of the spectral radius of a normal element. Corollaries: the uniqueness of the C^*-norm on a $*$-algebra; the automatic continuity of $*$-homomorphisms.
    - The main property of the spectrum of a selfadjoint element in a C^*-algebra. Corollaries: characters of a C^*-algebra are $*$-characters; the spectral invariance of C^*-subalgebras.
    - The 1st (commutative) Gelfand-Naimark theorem (unital case).

- The continuous functional calculus in C^*-algebras (definition, uniqueness, a necessary condition for the existence, the existence theorem).
- The spectral mapping theorem and the superposition property for the functional calculus.

4. **Spectral Theorem I: functional models**

- $*$-representations and $*$-modules. Cyclic $*$-modules. Examples.
- The functional model for a cyclic $*$-module over $C(X)$.
- A correspondence between normal operators and $*$-modules over $C(K)$ ($K \subset \mathbb{C}$). Relations between $*$-module morphisms and intertwining operators.
- $*$-cyclic operators and their relation to cyclic $*$-modules.
- The functional model for a $*$-cyclic normal operator.
- Hilbert direct sums of Hilbert spaces and of $*$-modules.
- Decomposing $*$-modules into Hilbert direct sums of cyclic submodules.
- The functional model for a $*$-module over $C(X)$.
- The functional model for a normal operator (Spectral Theorem I).

5. **Spectral Theorem II: Borel functional calculus**

- A correspondence between bounded linear operators on a Hilbert space and bounded sesquilinear forms.
- The weak measure topology and the weak operator topology.
- The WMT-density of $C(X)$ in $B(X)$.
- The separate continuity of the multiplication in $(B(X), \mathrm{WMT})$ and in $(\mathscr{B}(H), \mathrm{WOT})$.
- The canonical extension of representations of $C(X)$ to $B(X)$.
- The Borel functional calculus for a normal operator (definition, uniqueness, a necessary condition for the existence, the existence theorem – Spectral Theorem II).

6. **Spectral Theorem III: spectral measures**

- Orthogonal projections. An algebraic characterization of orthogonal projections. Orthogonal pairs of orthogonal projections (definition and algebraic characterizations). The norm of a linear combination of pairwise orthogonal projections.
- Finitely additive spectral measures and associated complex measures. Examples. The variation of a complex measure associated to a spectral measure.
- Integration of bounded measurable functions with respect to a finitely additive spectral measure. A 1-1 correspondence between finitely additive spectral measures and representations of $B_{\mathscr{A}}(X)$.
- Regular spectral measures on a compact Hausdorff topological space.
- A 1-1 correspondence between regular spectral measures and WMT-WOT-continuous representations of $B(X)$.
- A characterization of representations of $C(X)$ in terms of regular spectral measures.
- The spectral decomposition of a normal operator (Spectral Theorem III).