# Functional Analysis 2 (Operator Theory)

**Midterm exam 18.03.2022**

1. **Duality for operators on Banach spaces**
   - Annihilators and preannihilators.
   - The duals of subspaces and quotients.
   - Relations between the kernels and images of operators and of their duals.
   - A duality between injective operators and operators with dense image.
   - A duality between topologically injective and surjective operators.
   - A duality between isometries and coisometries.
   - The Closed Image Theorem.
   - Johnson's lemma on exact sequences of Banach spaces.
2. **Fredholm operators I: basic notions**
   - Fredholm operators. Fredholm index. Basic examples.
   - The additivity of the index.
   - Kato's lemma on the image of a Fredholm operator.
   - The fredholmness and the index of the dual operator.
3. **The Riesz–Schauder theory**
   - The ascent and the descent of a linear operator. Properties of linear operators with finite ascent and descent. The algebraic Riesz decomposition.
   - Riesz's theorem on operators "$\mathbf{1}$ + compact".
   - The Fredholm alternative. Abstract Fredholm theorems in Schauder's form.
   - Properties of the spectrum of a compact operator.
4. **Compact selfadjoint operators**
   - The adjoint of an operator between Hilbert spaces. A characterization of the adjoint operator in terms of inner products.
   - Basic properties of the operation $T \mapsto T^*$. The $C^*$-property.
   - Selfadjoint and normal operators. Properties of the spectrum and of eigenvectors of a self-adjoint operator.
   - The main property of the spectral radius of a normal operator.
   - A relation between invariant subspaces on an operator and of its adjoint. A corollary on invariant subspaces of a selfadjoint operator.
   - The Hilbert–Schmidt theorem.

5. **Fredholm operators II: an interplay between Fredholm and compact operators**

- Topological direct sums and complemented subspaces of normed spaces.
- Characterizations of complemented subspaces in terms of projections and in terms of quotients.
- Examples and counterexamples of complemented subspaces.
- The Nikolskii–Atkinson criterion for Fredholm operators.
- The Calkin algebra. The essential spectrum of a linear operator. The interpretation of the essential spectrum in terms of the Calkin algebra.
- The stability of the index under "small" perturbations.
- The stability of the index under compact perturbations.
- Nikolskii's characterization of Fredholm operators of index zero.
- Toeplitz operators on the Hardy space. The homomorphism $C(\mathbb{T}) \to \mathcal{Q}(H^2)$.
- The index formula for a Toeplitz operator with continuous symbol.

6. **Topological vector spaces**

- Topological vector spaces. The topology generated by a family of seminorms. Locally convex spaces and their "polynormability". Examples of locally convex spaces.
- A characterization of convergence in a locally convex space in terms of seminorms.
- A criterion for a locally convex space to be Hausdorff.
- A continuity criterion for a seminorm on a locally convex space.
- A continuity criterion for a linear operator between locally convex spaces.
- The domination relation for families of seminorms. Equivalent families of seminorms. Examples.
- Linear functionals on locally convex spaces (extension from subspaces, separation of points, separation of points and subspaces).