Fredholm operators

X, Y = vec. sp., $T: X \to Y$ linear.

Def. T is Fredholm $\Longleftrightarrow$ $\operatorname{Ker} T$ and $\operatorname{Coker} T$ are fin-dim.

$\operatorname{ind} T = \dim \operatorname{Ker} T - \dim \operatorname{Coker} T$ (index).

Thm. $X \xrightarrow{T} Y \xrightarrow{S} Z$

$$\boxed{\operatorname{ind}(ST) = \operatorname{ind} S + \operatorname{ind} T.}$$

Thm (Kato).
X, Y = Ban. sp, $T \in \mathcal{B}(X, Y)$, $\operatorname{Coker} T$ is fin-dim.
$\Longrightarrow$ $\operatorname{Im} T$ is closed in Y.

Proof. We may assume that $\operatorname{Ker} T = 0$.
(otherwise consider $\hat{T}: X/\operatorname{Ker} T \to Y$,
$\hat{T}(x + \operatorname{Ker} T) = T(x)$).

$Y = \operatorname{Im} T \oplus Y_0$, $\dim Y_0 < \infty$.

Consider $S: X \oplus_\infty Y_0 \to Y$, $S(x, y) = Tx + y$.

S is bijective and bounded $\xRightarrow[\text{(Ban.)}]{}$

$\Rightarrow S$ is a top. isom.

$X \oplus 0 \subset X \oplus_\infty Y$ is closed (exer) $\Longrightarrow$

$\Rightarrow S(X \oplus 0) = \operatorname{Im} T \subset Y$ is closed. $\square$

Thm. X, Y = Ban. sp, $T \in \mathscr{B}(X,Y)$. Then:

T is Fredholm $\iff$ T^* is Fredholm.

Moreover, if they are Fredholm, then

(1) $\operatorname{ind} T^* = -\operatorname{ind} T$;

(2) $\operatorname{ind} T = \dim \operatorname{Ker} T - \dim \operatorname{Ker} T^*$.

Proof. If either T or T^* is Fredholm $\xRightarrow{\text{(Kato)}}$ T satisfies the conditions of the closed image Thm. $\Rightarrow$ $\operatorname{Im} T$ and $\operatorname{Im} T^*$ are closed, and $\operatorname{Ker} T^* \cong (\operatorname{Coker} T)^*$, $\operatorname{Coker} T^* \cong (\operatorname{Ker} T)^*$.
$\Rightarrow$ both T and T^* are Fredholm, and (1), (2) hold. $\square$

The Riesz-Schauder theory

F. Riesz (1918)
J. Schauder (1930)

$$\boxed{\begin{array}{c}\mathbb{1} + K \text{ is Fredholm, and} \\ \operatorname{ind}(\mathbb{1}+K) = 0.\end{array}}$$

X = vec. sp., $T : X \to X$ linear.

Notation. $K_n = \operatorname{Ker} T^n$, $I_n = \operatorname{Im} T^n$ $(n \geq 0)$

$0 = K_0 \subset K_1 \subset K_2 \subset \ldots$; $X = I_0 \supset I_1 \supset I_2 \supset \ldots$

Def. The <u>ascent</u> of T is (подъём)

$a(T) = \min\{n : K_n = K_{n+j} \ \forall j \geq 0\}$

($a(T) = \infty$ if $\nexists$ such n)

The <u>descent</u> of T is (спуск)

$d(T) = \min\{n : I_n = I_{n+j} \ \forall j \geq 0\}$

($d(T) = \infty$ if $\nexists$ such n).

Lemma 1. (1) If $K_n = K_{n+1}$, then $K_n = K_{n+j} \ \forall j \geq 0$

(hence $a(T) < \infty$)

(2) If $I_n = I_{n+1}$, then $I_n = I_{n+j} \ \forall j \geq 0$

(hence $d(T) < \infty$).

(3) If $a(T) < \infty$ and $d(T) < \infty$, then $a(T) = d(T)$.

Proof. (1) It suff. to show that $K_{n+1} = K_{n+2}$.

Let $x \in K_{n+2}$; $T^{n+2}x = 0 = T^{n+1}(Tx) \Rightarrow$

$\Rightarrow Tx \in K_{n+1} = K_n \Rightarrow T^{n+1}x = T^n(Tx) = 0 \Rightarrow$

$\Rightarrow x \in K_{n+1}$.

(2) exer.

(3) Let's show that $a(T) \leq d(T)$.

Suppose that $a(T) > d(T)$; let $n = a(T) - 1$.

Then $K_n \subsetneq K_{n+1} = K_{n+2}$ and $I_n = I_{n+1}$.

Let $x \in K_{n+1}$; $T^n x \in I_n = I_{n+1} \Rightarrow T^n x = T^{n+1} y$

for some $y \in X \Rightarrow 0 = T^{n+1}x = T^{n+2}y \Rightarrow$

$\Rightarrow y \in K_{n+2} = K_{n+1} \Rightarrow 0 = T^{n+1}y = T^n x \Rightarrow x \in K_n$

$\Rightarrow K_{n+1} = K_n$, a contr.

Similarly, $d(T) \le a(T)$ (exer.). $\square$.

<u>Lemma 2.</u> (algebraic Riesz decomposition)

$T: X \to X$ linear; suppose $a(T) < \infty$, $d(T) < \infty$.

Let $p = a(T) = d(T)$, $K_p = \operatorname{Ker} T^p$, $I_p = \operatorname{Im} T^p$.

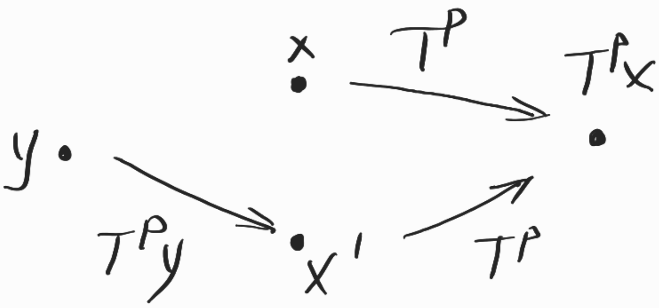
Then K_p and I_p are T-invariant,

$X = K_p \oplus I_p$, $T|_{K_p}$ is nilpotent,

$T|_{I_p} : I_p \to I_p$ is an isomorphism.

<u>Proof</u>. Clearly, K_p and I_p are T-inv, and $T|_{K_p}$ is nilpotent.

Let $x \in X \Rightarrow T^p x \in I_p = I_{2p} \Rightarrow T^p x = T^{2p} y$ for some $y \in X$.



Let $x' = T^p y$.

$T^p x' = T^{2p} y = T^p x$.

$\Rightarrow x - x' \in K_p \Rightarrow x = \underbrace{(x - x')}_{\in K_p} + \underbrace{x'}_{\in I_p} \in K_p + I_p$

$\Rightarrow X = K_p + I_p$.

Suppose $x \in K_p \cap I_p \Rightarrow T^p x = 0$ and $x = T^p z$ for some $z \in X \Rightarrow 0 = T^p x = T^{2p} z \Rightarrow$

$\Rightarrow z \in K_{2p} = K_p \Rightarrow T^p z = 0$, that is, $x = 0$.

$\Rightarrow K_p \cap I_p = 0 \Rightarrow X = K_p \oplus I_p$.

$T(I_p) = I_{p+1} = I_p \Rightarrow T|_{I_p} : I_p \to I_p$ is surj.

$K_p \cap I_p = 0$, and $\operatorname{Ker} T = K_1 \subset K_p \Rightarrow \operatorname{Ker} T \cap I_p = 0$

$\Rightarrow T|_{I_p}$ is inj. $\Rightarrow T|_{I_p} : I_p \to I_p$ is an isom. $\square$

<u>Thm.</u> (F. Riesz, 1918).

X = Ban. sp, $T \in \mathscr{B}(X)$, $T \in 1_X + \mathcal{K}(X)$. Then:

(1) T is Fredholm, and $\operatorname{ind} T = 0$.

(2) $a(T) < \infty$, $d(T) < \infty$.

(3) Let $p = a(T) = d(T)$, $K_p = \operatorname{Ker} T^p$, $I_p = \operatorname{Im} T^p$.
Then K_p and I_p are closed T-inv. subspaces,
$X = K_p \oplus I_p$, K_p is fin-dim, $T|_{K_p}$ is nilpotent,
$T|_{I_p} : I_p \to I_p$ is a topol. isomorphism.

<u>Proof.</u> Let $T = 1 - S$ where $S \in \mathcal{K}(X)$.

<u>Step 1:</u> $\operatorname{Ker} T$ is fin-dim, and $\operatorname{Im} T$ is closed.

<u>Proof.</u> Let $K = \operatorname{Ker} T \Rightarrow S|_K = 1$; $S|_K$ is comp.
$\Rightarrow K$ is fin-dim.

Consider $\hat{T} : X/K \to X$, $\hat{T}(x+K) = Tx$.

It suff. to show that $\hat{T}$ is top. injective,
that is, $\exists c > 0$ s.t. $\|\hat{T}u\| \geq c \quad \forall u \in X/K$, $\|u\| = 1$.

Assume there is no such $c>0$

$\Rightarrow \exists$ a seq (u_n), $u_n \in X/K$, $\|u_n\|=1$, s.t. $\hat{T}u_n \to 0$ $(n\to\infty)$.

Let $u_n = x_n + K$, $x_n \in X$, $\|x_n\| \le 2$. $\Rightarrow$

$\Rightarrow$ dist$(x_n, K) = \|u_n\| = 1$, $Tx_n \to 0$.

We may assume that $\underset{x_n - Tx_n}{\underset{\parallel}{Sx_n}} \to x \in X$ (because S is comp).

$\Rightarrow x_n \to x \Rightarrow Tx_n \to Tx \Rightarrow Tx = 0$; that is, $x \in K$.

On the other hand, dist$(x_n, K) = 1$, $x_n \to x$

This is a contr. $\Rightarrow$ ImT is closed. $\square$.

<u>Step 2</u> T is Fredholm.

<u>Proof</u>. ImT is closed $\Rightarrow$ Coker$T$ is a Ban. sp;

The Cl. Im. Thm $\Rightarrow (\mathrm{Coker}T)^* \cong \mathrm{Ker}T^*$.

$T^* = 1 - S^* \in 1 + \mathcal{K}(X^*) \Rightarrow \mathrm{Ker}T^*$ is fin-dim. $\Rightarrow$

$\Rightarrow$ so is Coker$T$ $\Rightarrow$ T is Fredholm. $\square$.

<u>Step 3</u> $a(T) < \infty$.

<u>Proof</u>. Assume $a(T) = \infty \Rightarrow K_n \subsetneq K_{n+1}$ $\forall n$

Choose $x_n \in K_{n+1}$ which is a $1/2$-$\perp$ to K_n

(that is, $\|x_n\| = 1$ and dist$(x_n, K_n) \ge 1/2$)

Then $\forall n, m \in \mathbb{N}$, $n \ge m+1$, we have

$\|Sx_n - Sx_m\| = \|(1-T)x_n - (1-T)x_m\| =$

$$= \| x_n - \underbrace{Tx_n}_{\in K_n} + \underbrace{x_m - Tx_m}_{\in K_{m+1} \subset K_n} \| \geq 1/2 \text{, a contr.}$$

with the compactness of S. $\square$

<u>Step 4.</u> $d(T) < \infty$.

<u>Proof.</u> We know: $a(T^*) < \infty \Rightarrow$

$\Rightarrow \exists\, n \geq 0$ s.t. $\operatorname{Ker}(T^*)^n = \operatorname{Ker}(T^*)^{n+1}$.

$$\Rightarrow \quad {}^{\perp}\operatorname{Ker}(T^n)^* = {}^{\perp}\operatorname{Ker}(T^{n+1})^*$$

$$\begin{array}{ccc} \| & & \| \\ \overline{\operatorname{Im}(T^n)} & & \overline{\operatorname{Im}(T^{n+1})} \\ \| & & \| \\ \operatorname{Im}(T^n) & & \operatorname{Im}(T^{n+1}) \end{array}$$

(because T^k is Fred. $\forall k \geq 0$) $\underset{L1}{\Longrightarrow} d(T) < \infty$.

<u>Step 5.</u> T is Fred $\Rightarrow$ so is $T^p \Rightarrow$

$\Rightarrow K_p$ is fin-dim, and I_p is closed.

Now L2 implies (3).

(by the Banach Inv. Map Thm).

$$\Rightarrow \operatorname{ind} T = \underbrace{\operatorname{ind}(T|_{K_p})}_{\substack{=0 \\ (K_p \text{ is fin-dim})}} + \underbrace{\operatorname{ind}(T|_{I_p})}_{=0} = 0. \qquad \square\square$$

Cor. 1. (Fredholm Alternative)

$T \in 1_X + \mathcal{K}(X)$ is inj. $\Longleftrightarrow$ T is surj. $\Longleftrightarrow$

$\Longleftrightarrow$ T is bijective.

Cor. 2. (J. Schauder 1930)

(abstract Fredholm theorems)

$T \in 1_X + \mathcal{K}(X)$. Then

(1) $\dim \operatorname{Ker} T = \dim \operatorname{Ker} T^* < \infty$.

(2) $\operatorname{Im} T = {}^{\perp}(\operatorname{Ker} T^*)$

(3) $\operatorname{Im} T^* = (\operatorname{Ker} T)^{\perp}$.