Thm. 1. X, Y = Banach sp, $T \in \mathcal{B}(X,Y)$

(1) T is top. inj. $\Longleftrightarrow$ T^* is surj.

(2) T is surj. $\Longleftrightarrow$ T^* is top. inj.

(3) T is a top. isom. $\Longleftrightarrow$ T^* is a top. isom.

Thm 2. X, Y = Ban. sp, $T \in \mathcal{B}(X,Y)$

(1) T is an isometry $\Longleftrightarrow$ T^* is a coisometry.

(2) T is a coisometry $\Longleftrightarrow$ T^* is an isometry.

(3) T is an isometric isom $\Longleftrightarrow$ T^* is an isometric iso.

Proof: Exer (similar to the pf of Thm 1).

Def. X, Y = vector spaces, $T : X \to Y$ linear.
The cokernel of T is $\operatorname{Coker} T = Y / \operatorname{Im} T$.

Thm 3. X, Y = Banach sp, $T \in \mathcal{B}(X,Y)$ TFAE:

(1) $\operatorname{Im} T$ is closed in Y; (2) $\operatorname{Im} T^*$ is closed in X^*;

(3) $\operatorname{Im} T = {}^{\perp}(\operatorname{Ker} T^*)$; (4) $\operatorname{Im} T^* = (\operatorname{Ker} T)^{\perp}$.

Moreover, if they are satisfied, then $\exists$ isometric isom.

$\operatorname{Ker} T^* \cong (\operatorname{Coker} T)^*$ and $\operatorname{Coker} T^* \cong (\operatorname{Ker} T)^*$.

(the Closed Image Thm).

Proof. (1)$\Longleftrightarrow$(3) clear, because

$$\overline{\operatorname{Im} T} = {}^{\perp}(\operatorname{Ker} T^*) \quad \forall T.$$

(1) $\Rightarrow$ (4)

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, "Q"'] & Y \\ X/\operatorname{Ker}T \arrow[ur, "T_0"'] & \end{tikzcd} \qquad \begin{tikzcd} X^* & Y^* \arrow[l, "T^*"'] \arrow[dl, "T_0^*"] \\ (X/\operatorname{Ker}T)^* \arrow[u, "Q^*"] & \end{tikzcd}$$

$\operatorname{Im}T_0 = \operatorname{Im}T$ is closed ; T_0 is injective.

OMT $\Rightarrow$ T_0 is topol. inj. $\Rightarrow$ T_0^* is surj.

$\Rightarrow \operatorname{Im}T^* = \operatorname{Im}Q^* = (\operatorname{Ker}T)^\perp$ (see prev. lec.)

(4) $\Rightarrow$ (2) clear.

(2) $\Rightarrow$ (1)

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[dr, "T_0"'] & Y \\ & \overline{\operatorname{Im}T} \arrow[u, "J"'] \end{tikzcd} \qquad \begin{tikzcd} X^* & Y^* \arrow[l, "T^*"'] \arrow[d, "J^*"] \\ & (\overline{\operatorname{Im}T})^* \arrow[ul, "T_0^*"] \end{tikzcd}$$

J = inclusion $\Rightarrow$ J^* = restr. map

Hahn-Ban. $\Rightarrow$ J^* is surj. $\Rightarrow$

$\Rightarrow \operatorname{Im}T_0^* = \operatorname{Im}T^*$ is closed

T_0 has dense image $\Rightarrow$ T_0^* is inj.

$\Rightarrow$ T_0^* is topol. inj. (by OMT) $\Rightarrow$ T_0 is surj.

$\Rightarrow \operatorname{Im}T = \overline{\operatorname{Im}T}$.

Now suppose that (1)–(4) are satisfied.

$(\operatorname{Coker}T)^* \cong (Y/\operatorname{Im}T)^* \cong (\operatorname{Im}T)^\perp = \operatorname{Ker}T^*$.

$$(\mathrm{Ker}\,T)^* \cong X^*/(\mathrm{Ker}\,T)^\perp = X^*/\mathrm{Im}\,T^* = \mathrm{Coker}\,T^*.$$

$\square$

<u>Cor.</u> (Johnson's lemma)
X, Y, Z = Ban. sp., $T \in \mathcal{B}(X,Y)$, $S \in \mathcal{B}(Y,Z)$
TFAE:

(1) $X \xrightarrow{T} Y \xrightarrow{S} Z$ is exact, and $\mathrm{Im}\,S$ is closed.

(2) $Z^* \xrightarrow[S^*]{} Y^* \xrightarrow[T^*]{} X^*$ is exact, and $\mathrm{Im}\,T^*$ is closed.

<u>Proof.</u> (1) $\Rightarrow$ (2)

$\mathrm{Ker}\,T^* = (\mathrm{Im}\,T)^\perp = (\mathrm{Ker}\,S)^\perp = \mathrm{Im}\,S^*$;

$\mathrm{Im}\,T^*$ is closed because $\mathrm{Im}\,T = \mathrm{Ker}\,S$ is closed.

(2) $\Rightarrow$ (1)

$\mathrm{Ker}\,S = {}^\perp(\mathrm{Im}\,S^*) = {}^\perp(\mathrm{Ker}\,T^*) = \mathrm{Im}\,T$

(because $\mathrm{Im}\,T$ is closed, see Thm 3)

$\mathrm{Im}\,S$ is closed because $\mathrm{Im}\,S^* = \mathrm{Ker}\,T^*$ is closed. $\square$.

<u>Cor.</u> A chain complex $C = (C_n, d_n)_{n\in\mathbb{Z}}$ of Ban. spaces is exact $\iff$ $C^* = (C_n^*, d_n^*)$ is exact.

Fredholm operators

X, Y = vector spaces, $T : X \to Y$ linear.

<u>Def.</u> T is a <u>Fredholm operator</u> $\iff$ $\mathrm{Ker}\,T$ and $\mathrm{Coker}\,T$ are finite-dimensional.

If X, Y are Banach sp., then T is assumed to be bdd.

<u>Def.</u> The <u>index</u> of a Fredholm oper. T is
$\operatorname{ind} T = \dim \operatorname{Ker} T - \dim \operatorname{Coker} T$.

<u>Example 1.</u> Any isomorphism $T: X \to Y$ is Fredholm, and $\operatorname{ind} T = 0$.

<u>Example 2.</u> $P: X \to X$ a projection
(that is, $X = X_0 \oplus X_1$, $P(x_0 + x_1) = x_1$ $\forall x_0 \in X_0, x_1 \in X_1$)
suppose $\dim X_0 < \infty$.
Then P is Fredholm, and $\operatorname{ind} P = 0$.

<u>Example 3.</u> X, Y = fin.-dim. spaces $\Longrightarrow$
$\Rightarrow$ any lin. $T: X \to Y$ is Fredholm, and
$\operatorname{ind} T = \dim X - \dim Y$. Indeed,

$$\operatorname{ind} T = \dim \operatorname{Ker} T - \dim \operatorname{Coker} T =$$
$$= \dim \operatorname{Ker} T - (\dim Y - \dim \operatorname{Im} T) =$$
$$= \underbrace{\dim \operatorname{Ker} T + \dim \operatorname{Im} T}_{= \dim X} - \dim Y.$$

In particular, if $T: X \to X$, then $\operatorname{ind} T = 0$.

<u>Example 4.</u> $X = \ell^p$ $(1 \le p \le \infty)$ or $X = c_0$;
the left and right shift ops $T_\ell: X \to X$,
$T_r: X \to X$ are Fredholm, $\operatorname{ind} T_\ell = 1$, $\operatorname{ind} T_r = -1$.

Thm. X, Y, Z = vec. spaces, $T: X \to Y$, $S: Y \to Z$ linear.

(1) If any two of S, T, ST are Fredholm, then so is the third one.

(2) If they are Fredholm, then $\boxed{\operatorname{ind}(ST) = \operatorname{ind} S + \operatorname{ind} T.}$

(3) If ST is Fredholm, then $\operatorname{Ker} T$ and $\operatorname{Coker} S$ are fin-dim.

Lemma 1. Suppose $0 \to X_0 \to X_1 \to \cdots \to X_n \to 0$ is an exact seq. of fin-dim. vec. spaces Then $\sum (-1)^i \dim X_i = 0$

Proof: exer. $\square$.

Lemma 2. (Snake Lemma)

Let

$$\begin{array}{ccccccccc}
0 & \to & X_1 & \to & X_2 & \to & X_3 & \to & 0 \\
 & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\
0 & \to & Y_1 & \to & Y_2 & \to & Y_3 & \to & 0
\end{array}$$

be a comm. diag. of vec. spaces with exact rows. Then $\exists$ an exact seq.

$$0 \to \operatorname{Ker}\alpha \to \operatorname{Ker}\beta \to \operatorname{Ker}\gamma \to \operatorname{Coker}\alpha \to \operatorname{Coker}\beta \to \operatorname{Coker}\gamma \to 0.$$

Proof of Thm. Consider the diag

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "\binom{1}{T}"] \arrow[d, "T"'] & X \oplus Y \arrow[r, "(-T\ 1)"] \arrow[d, "ST \oplus 1"] & Y \arrow[r] \arrow[d, "S"] & 0 \\
0 \arrow[r] & Y \arrow[r, "\binom{S}{1}"'] & Z \oplus Y \arrow[r, "(-1\ S)"'] & Z \arrow[r] & 0
\end{tikzcd}$$

It is comm and has exact rows (exer.)

L2 $\Rightarrow$ $\exists$ an exact seq.

$$0 \to \overset{(+)}{\mathrm{Ker}\,T} \to \overset{(-)}{\mathrm{Ker}(ST \oplus 1)} \to \overset{(+)}{\mathrm{Ker}\,S} \to$$
$$\cong \mathrm{Ker}(ST)$$

$$\to \overset{(-)}{\mathrm{Coker}\,T} \to \overset{(+)}{\mathrm{Coker}(ST \oplus 1)} \to \overset{(-)}{\mathrm{Coker}\,S} \to 0$$
$$\cong \mathrm{Coker}\,ST$$

This implies (1), (3).

L1 $\Rightarrow$ $\mathrm{ind}\,S - \mathrm{ind}(ST) + \mathrm{ind}\,T = 0$. $\square$.