Commutative Banach algebras and C^*-algebras. Spectral theory of operators on a Hilbert space

Exercises marked by "**-B**" are optional. If you solve such exercises, you will earn bonus points.

4.1. Recall that $C^n[0,1]$ is a Banach algebra under the norm $\|f\| = \sum_{k=0}^n \frac{\|f^{(k)}\|_\infty}{k!}$ (where $\|\cdot\|_\infty$ is the supremum norm). Describe the maximal spectrum and the Gelfand transform of $C^n[0,1]$.

4.2. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. The *disk algebra* $\mathscr{A}(\bar{\mathbb{D}})$ consists of those $f \in C(\bar{\mathbb{D}})$ that are holomorphic on $\mathbb{D}$. Show that $\mathscr{A}(\bar{\mathbb{D}})$ is a closed subalgebra of $C(\bar{\mathbb{D}})$. Describe the maximal spectrum and the Gelfand transform of $\mathscr{A}(\bar{\mathbb{D}})$.

4.3. Let $f, g \in \ell^1(\mathbb{Z})$. The *convolution* of f and g is the function $f * g$ on $\mathbb{Z}$ given by

$$(f * g)(n) = \sum_{k \in \mathbb{Z}} f(k) g(n-k) \qquad (n \in \mathbb{Z}). \tag{1}$$

(a) Show that the series in (1) converges, that $f * g \in \ell^1(\mathbb{Z})$, and that $\ell^1(\mathbb{Z})$ is a commutative unital Banach algebra under convolution.
(b) Show that $\ell^1(\mathbb{Z})$ contains the group algebra $\mathbb{C}\mathbb{Z}$ as a dense subalgebra.
(c) Describe the maximal spectrum and the Gelfand transform of $\ell^1(\mathbb{Z})$.
Hint to **(c)**: each character χ of $\ell^1(\mathbb{Z})$ is uniquely determined by $\chi(\delta_1) \in \mathbb{C}$, where δ_1 is the element of $\mathbb{C}\mathbb{Z}$ corresponding to $1 \in \mathbb{Z}$. Show that, if $\chi \neq 0$, then $\chi(\delta_1) \in \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.

4.4-B. Extend Exercise 4.3 to $\ell^1(G)$, where G is any (discrete) abelian group. (*Hint:* the maximal spectrum of $\ell^1(G)$ is homeomorphic to the *dual group* $\widehat{G}$ of G, which consists of all group homomorphisms from G to $\mathbb{T}$.)

4.5. Show that
(a) $C^n[0,1]$ is a Banach $*$-algebra under the involution $f^*(t) = \overline{f(t)}$ ($t \in [0,1]$), but is not a C^*-algebra unless $n = 0$;
(b) $\mathscr{A}(\bar{\mathbb{D}})$ is a Banach $*$-algebra under the involution $f^*(z) = \overline{f(\bar{z})}$ ($z \in \bar{\mathbb{D}}$), but is not a C^*-algebra;
(c) $\ell^1(\mathbb{Z})$ is a Banach $*$-algebra under the involution $f^*(n) = \overline{f(-n)}$ ($n \in \mathbb{Z}$), but is not a C^*-algebra.

4.6-B. (a) Does there exist a norm and an involution on $C^1[a,b]$ making it into a C^*-algebra?
(b) Does there exist a norm and an involution on $\mathscr{A}(\bar{\mathbb{D}})$ making it into a C^*-algebra?
(c) Does there exist a norm and an involution on $\ell^1(\mathbb{Z})$ making it into a C^*-algebra?
Remark. In 3.8 (a,b,c), we do not assume that the new norm is equivalent to the original norm.

4.7. Let $\alpha \in \ell^\infty$, and let M_α denote the respective diagonal operator on ℓ^2. Show that for each $f \in C(\sigma(M_\alpha))$ we have $f(M_\alpha) = M_{f \circ \alpha}$.

4.8. Let (X, μ) be a σ-finite measure space, let $\varphi \colon X \to \mathbb{C}$ be an essentially bounded measurable function, and let M_φ denote the respective multiplication operator on $L^2(X, \mu)$. Show that for each $f \in C(\sigma(M_\varphi))$ we have $f(M_\varphi) = M_{f \circ \varphi}$ (in particular, give a precise meaning to the expression $f \circ \varphi$).

4.9. Extend the results of Exercises 4.7 and 4.8 to the Borel functional calculus.

4.10. Let A be a unital C^*-algebra, and let $u \in A$ be a unitary element. Show that $\sigma(u) \subset \mathbb{T}$ (where $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$).

4.11. Let A be a unital C^*-algebra.
(a) Prove that for each selfadjoint element $a \in A$ the element $u = \exp(ia)$ is unitary.
(b) Prove that if $u \in A$ is a unitary element such that $\sigma(u) \neq \mathbb{T}$, then there exists a selfadjoint element $a \in A$ such that $u = \exp(ia)$.
(c) Does (b) hold if $\sigma(u) = \mathbb{T}$?

4.12. Show that for each unitary operator U on a Hilbert space H there exists a bounded selfadjoint operator T on H such that $U = \exp(iT)$ (compare with Exercise 4.11 (c)).

Hint: the function $[0, 2\pi) \to \mathbb{T}$, $t \mapsto \exp(it)$, is a Borel bijection.

4.13. Show that a compact selfadjoint operator T is cyclic **(a)** if and **(b)** only if all the eigenvalues of T have multiplicity 1.

4.14. Let $\varphi\colon [a, b] \to \mathbb{R}$ be a strictly monotone, continuous function. Prove that the multiplication operator $M_\varphi\colon L^2[a, b] \to L^2[a, b]$ is cyclic.

4.15. Let T denote the operator on $L^2[0, 1]$ defined by $(Tf)(t) = \sqrt{t}f(t)$. Find explicitly a positive Radon measure μ on $[0, 1]$ and a unitary isomorphism $U\colon L^2[0, 1] \to L^2([0, 1], \mu)$ which establishes a unitary equivalence between T and the multiplication operator M_t given by $(M_t f)(t) = tf(t)$.

4.16. **(a)** Show that a bounded operator P on a Hilbert space is an orthogonal projection if and only if $P = P^* = P^2$.
(b) By using the spectral theorem, prove that a bounded normal operator P such that $\sigma(P) \subset \{0, 1\}$ is an orthogonal projection.
(c) Prove (b) without using the spectral theorem.

4.17-B. Let T be a cyclic selfadjoint operator on a Hilbert space H. Prove that an operator $S \in \mathscr{B}(H)$ commutes with T if and only if there exists a bounded Borel function $f\colon \sigma(T) \to \mathbb{C}$ such that $S = f(T)$.

4.18-B. Let H be an infinite-dimensional separable Hilbert space. Prove that $\mathscr{K}(H)$ is a unique closed two-sided ideal of $\mathscr{B}(H)$ different from 0 and $\mathscr{B}(H)$.

Hint. Let $0 \neq I \subset \mathscr{B}(H)$ be a two-sided ideal. Recall the standard proof of the simplicity of the matrix algebra $M_n(\mathbb{C})$, and apply the same argument to show that I contains all finite rank operators. If I contains at least one noncompact operator, apply the spectral theorem.