Fredholm and compact operators

2.1. Let $0 \to X^0 \to X^1 \to \ldots \to X^n \to 0$ be an exact sequence of finite-dimensional vector spaces. Show that $\sum_i (-1)^i \dim X^i = 0$. (We used this result in our proof of the additivity of the index, see the lectures.)

2.2. What can you say about an operator that is compact and Fredholm simultaneously?

2.3. Let $a_0, \ldots, a_n \in C^p[a,b]$. Show that the operator

$$D\colon C^{p+n}[a,b] \to C^p[a,b], \quad D(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y,$$

is Fredholm, and find the index of D.

2.4. Show that the operator $D\colon C^1(S^1) \to C(S^1)$, $D(f) = f'$, is Fredholm, and find the index of D.

2.5. Let $\lambda \in \ell^\infty$, and let M_λ denote the respective diagonal operator on ℓ^p or on c_0. Find a condition on λ that is necessary and sufficient for M_λ to be Fredholm. Find the index of M_λ.

2.6. Let $f \in C[a,b]$, and let M_f denote the multiplication operator by f on $C[a,b]$. Find a condition on f that is necessary and sufficient for M_f to be Fredholm. Find the index of M_f.

2.7. Let $I \subset \mathbb{R}$ be an interval (i.e., any connected set), let $f\colon I \to \mathbb{C}$ be an essentially bounded measurable function, and let M_f denote the multiplication operator by f on $L^p(I)$ $(1 \leqslant p \leqslant \infty)$. Find a condition on f that is necessary and sufficient for M_f to be Fredholm. Find the index of M_f.

2.8. Let H be an infinite-dimensional Hilbert space. Show that for each $n \in \mathbb{Z}$ there exists a Fredholm operator of index n on H.

2.9. Let X be a Banach space. Suppose that $T \in \mathscr{B}(X)$, $K \in \mathscr{K}(X)$, and let $S = T + K$.
(a) Prove that, if $\lambda \in \sigma(T)$ is not an eigenvalue of T of finite multiplicity, then $\lambda \in \sigma(S)$.
(b) Show that, if T is the shift operator on $\ell^2(\mathbb{Z})$, then we can find K in such a way that $\sigma(S) = \{z \in \mathbb{C} : |z| \leqslant 1\}$ (while $\sigma(T) = \{z \in \mathbb{C} : |z| = 1\}$).

2.10 (*the classical Fredholm theorems*). Let $I = [a,b]$. Define a bilinear form on $C(I)$ by $\langle f, g\rangle = \int_a^b fg\,dt$. For each $K \in C(I \times I)$ define $K' \in C(I \times I)$ by $K'(x,y) = K(y,x)$. Let $T_K\colon C(I) \to C(I)$ denote the integral operator given by

$$(T_K f)(x) = \int_a^b K(x,y) f(y)\,dy.$$

Let $S_K = \mathbf{1} - T_K$. Prove that
(a) $f \in \operatorname{Im} S_K \iff \langle f, g\rangle = 0 \quad \forall g \in \operatorname{Ker} S_{K'}$;
(b) $\dim \operatorname{Ker} S_K = \dim \operatorname{Ker} S_{K'} < \infty$.
Hint. Extend S_K and $S_{K'}$ to $L^2(I)$ and apply the Fredholm theorems in Schauder's form (see the lectures).

2.11. Describe explicitly the (Hilbert) adjoints of the following operators. Determine whether they are selfadjoint and whether they are (except for (f)) normal.
(a) the diagonal operator M_λ on ℓ^2 defined by $(x_1, x_2, \ldots) \mapsto (\lambda_1 x_1, \lambda_2 x_2, \ldots)$ $(\lambda \in \ell^\infty,\ x \in \ell^2)$;
(b) the multiplication operator M_f on $L^2(X,\mu)$ defined by $g \mapsto fg$ $(f \in L^\infty(X,\mu),\ g \in L^2(X,\mu))$;
(c) the right shift and the left shift operators on ℓ^2;
(d) the bilateral shift operator T_b on $\ell^2(\mathbb{Z})$ defined by $(T_b(x))_i = x_{i-1}$ $(i \in \mathbb{Z})$;
(e) the Volterra operator of "taking the primitive" on $L^2[0,1]$ (see Exercise 1.7);
(f) the Hilbert–Schmidt integral operator on $L^2(X,\mu)$ (see Exercise 1.9).

2.12. Calculate the norm of the Volterra operator T from Exercise 2.11 (e).
Hint: T^*T is compact and selfadjoint.

2.13. For each of the following operators T find $\sigma_{\mathrm{ess}}(T)$ and calculate $\operatorname{ind}(T - \lambda\mathbf{1})$ for each $\lambda \in \mathbb{C} \setminus \sigma_{\mathrm{ess}}(T)$: **(a)** the diagonal operator on ℓ^p or on c_0; **(b)** the multiplication operator by a continuous function on $C[a, b]$ or by a bounded measurable function on $L^p[a, b]$; **(c)** the left shift on ℓ^p or on c_0; **(d)** the right shift on ℓ^p or on c_0; **(e)** the bilateral shift on $\ell^2(\mathbb{Z})$; **(f)** an arbitrary compact operator.

2.14. (a) Let H be a Hilbert space. Let us take for granted the fact that the group $\mathrm{GL}(H)$ of invertible bounded operators on H is path connected[1] (we will prove this fairly soon). Show that two Fredholm operators $S, T \in \mathscr{B}(H)$ belong to the same connected component of $\mathsf{Fred}(H)$ $\iff$ there is a continuous path in $\mathsf{Fred}(H)$ connecting S and T $\iff$ $\operatorname{ind} S = \operatorname{ind} T$.
(b) Let H be an infinite-dimensional Hilbert space, and let $\mathcal{Q}(H) = \mathscr{B}(H)/\mathscr{K}(H)$ be the Calkin algebra. Let G denote the group of invertibles in $\mathcal{Q}(H)$, and let $G_0 \subset G$ denote the connected component of the identity. Prove that the index induces a group isomorphism $G/G_0 \cong \mathbb{Z}$.

2.15-B. Let $f \in C(\mathbb{T})$, and let T_f denote the corresponding Toeplitz operator on the Hardy space $H^2(\mathbb{T})$. Recall (see the lectures) that, if $f(z) \neq 0$ for each $z \in \mathbb{T}$, then T_f is Fredholm.
(a) Suppose that $f(z) = 0$ for some $z \in \mathbb{T}$. Prove that T_f is not Fredholm.
(b) Find $\sigma_{\mathrm{ess}}(T_f)$ in terms of f.
(c) Find $\|T_f\|$ in terms of f.

2.16-B. Prove that c_0 is not complemented in ℓ^∞.
Hint. Use the following plan:
1) Prove that $\mathbb{N}$ is the union of an uncountable family of countable sets A_i such that $A_i \cap A_j$ is finite for all $i \neq j$. (Hint: it is convenient to replace $\mathbb{N}$ by $\mathbb{Q}$).
2) Prove that if $f \in (\ell^\infty)^*$ vanishes on c_0, then the set of those $i \in I$ for which $f(\chi_{A_i}) \neq 0$ is at most countable.
3) Deduce that c_0 is not complemented in ℓ^∞.

[1] In fact, if H is infinite-dimensional, then $\mathrm{GL}(H)$ is contractible (Kuiper's theorem).