# Compact operators

Exercises marked by "**-B**" are optional. If you solve such exercises, you will earn bonus points.

**1.1.** Let $X$ be a normed space, let $f \in X^* \setminus \{0\}$, and let $X_0 = \mathrm{Ker}\, f$. Show that there exists a 0-perpendicular to $X_0$ in $X$ iff $f$ is norm-attaining (which means that there exists $x \in X$, $\|x\| = 1$, such that $|f(x)| = \|f\|$). Give an example showing that this is not always the case.

**1.2. (a)** Prove that a subset $S \subset c_0$ is relatively compact iff there exists $y \in c_0$ such that $|x_n| \leqslant |y_n|$ for all $x \in S$ and all $n \in \mathbb{N}$.      **(b)** Does a similar result hold for $\ell^p$?

**1.3.** Are the left and right shift operators on $\ell^p$ and on $c_0$ compact?

**1.4.** Can the image of a compact operator between Banach spaces contain an infinite-dimensional closed vector subspace?

**1.5.** Prove that the inclusion $C^1[a, b] \to C[a, b]$ is a compact operator.

**1.6. (a)** Let $f \in C[a, b]$, and let $M_f$ denote the respective multiplication operator on $C[a, b]$. Find a condition on $f$ that is necessary and sufficient for $M_f$ to be compact.
**(b)** Let $I \subset \mathbb{R}$ be an interval (not necessarily open or closed, not necessarily bounded), let $f \colon I \to \mathbb{C}$ be an essentially bounded measurable function, and let $M_f$ denote the respective multiplication operator on $L^p(I)$ ($1 \leqslant p \leqslant \infty$). Find a condition on $f$ that is necessary and sufficient for $M_f$ to be compact.

**1.7.** Given an integrable function $f$ on $[0, 1]$, define a function $Tf$ on $[0, 1]$ by

$$(Tf)(x) = \int_0^x f(t)\, dt.$$

Is $T$ a compact operator    **(a)** from $C[0, 1]$ to $C[0, 1]$?      **(b)** from $L^p[0, 1]$ to $C[0, 1]$ (where $1 < p \leqslant \infty$)?    **(c)** from $L^p[0, 1]$ to $L^p[0, 1]$ (where $1 < p \leqslant \infty$)?    **(d)** from $L^1[0, 1]$ to $C[0, 1]$? **(e)** from $L^1[0, 1]$ to $L^1[0, 1]$?

**1.8.** Let $I = [a, b]$, and let $K \in C(I \times I)$. Prove that the integral operator $T \colon C(I) \to C(I)$,

$$(Tf)(x) = \int_a^b K(x, y) f(y)\, dy,$$

is compact.

**1.9.** Let $(X, \mu)$ be a measure space, and let $K \in L^2(X \times X, \mu \times \mu)$. Prove that the *Hilbert–Schmidt integral operator* $T_K \colon L^2(X, \mu) \to L^2(X, \mu)$,

$$(T_K f)(x) = \int_X K(x, y) f(y)\, d\mu(y),$$

is compact.

     *Hint:* show that functions of the form $K(x, y) = f(x) g(y)$, where $f, g \in L^2(X, \mu)$, span a dense subspace of $L^2(X \times X, \mu \times \mu)$, and use the fact that $\|T_K\| \leqslant \|K\|_2$ (see Exercise 2.8).

**1.10. (a)** Let $X$ be a compact metrizable topological space, let $K \in C(X \times X)$, and let $\mu$ be a finite Borel measure on $X$. Show that the image of the Hilbert–Schmidt integral operator $T_K \colon L^2(X, \mu) \to L^2(X, \mu)$ is contained in $C(X)$, and that $T_K$ is a compact operator from $L^2(X, \mu)$ to $C(X)$.
**(b)-B** Extend (a) to an arbitrary (not necessarily metrizable) compact topological space $X$.