1. **Topological vector spaces**

- A characterization of convergence in a locally convex space in terms of seminorms.
- A criterion for a locally convex space to be Hausdorff.
- A continuity criterion for a seminorm on a locally convex space.
- A continuity criterion for a linear operator between locally convex spaces.
- The domination relation for families of seminorms. Equivalent families of seminorms. Examples.
- Linear functionals on locally convex spaces (extension from subspaces, separation of points, separation of points and subspaces).
- Dual pairs of vector spaces. The weak topology of a dual pair. Special cases and basic properties of the weak topology.
- A characterization of linear functionals that are continuous with respect to the weak topology.
- A characterization of reflexive Banach spaces in terms of topologies on the dual.
- Dual operators between dual pairs of vector spaces. An existence criterion for the dual operator in terms of weak continuity.
- Relations between the continuity, the weak continuity, and the existence of the dual for an operator between locally convex spaces (in particular, between normed spaces).
- Annihilators. Basic properties of annihilators. The double annihilator theorem (for dual pairs and for locally convex spaces). Corollaries: the weak closure of a vector subspace; a density criterion for a vector subspace; relations between kernels and images of a linear operator and of the dual operator.
- Equicontinuous families of linear operators. The Banach–Alaoglu–Bourbaki theorem.

2. **Commutative Banach algebras**

- The maximal spectrum and the character space of a commutative unital algebra.
- The closedness of maximal ideals of a commutative unital Banach algebra.
- The 1-1 correspondence between the character space and the maximal spectrum of a commutative unital Banach algebra.
- An invertibility criterion in terms of characters.
- The Gelfand topology on the maximal spectrum.
- The compactness of the maximal spectrum (unital case).
- The Gelfand transform of a commutative unital Banach algebra.
- Properties of the Gelfand transform.
- The maximal spectrum and the Gelfand transform for subalgebras of $C(X)$. 

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3. $C^*$-algebras and continuous functional calculus

- Selfadjoint elements, unitary elements, normal elements.
- The main property of the spectral radius of a normal element. Corollaries: the uniqueness of the $C^*$-norm on a $*$-algebra; the automatic continuity of $*$-homomorphisms.
- The main property of the spectrum of a selfadjoint element in a $C^*$-algebra. Corollaries: characters of a $C^*$-algebra are $*$-characters; the spectral invariance of $C^*$-subalgebras.
- The 1st (commutative) Gelfand-Naimark theorem (unital case).
- The continuous functional calculus in $C^*$-algebras (definition, a necessary condition for the existence, the existence theorem).
- The spectral mapping theorem and the superposition property for the functional calculus.

4. Spectral Theorem I: functional models

- The functional model for a cyclic $*$-module over $C(X)$.
- A correspondence between normal operators and $*$-modules over $C(K)$ ($K \subset \mathbb{C}$).
- $*$-cyclic operators and their relation to cyclic $*$-modules.
- The functional model for a $*$-cyclic normal operator.
- Hilbert direct sums of Hilbert spaces and of $*$-modules.
- A decomposition of a $*$-module into a Hilbert direct sum of cyclic submodules.
- The functional model for a $*$-module over $C(X)$.
- The functional model for a normal operator (Spectral Theorem I).

5. Spectral Theorem II: Borel functional calculus

- A correspondence between bounded linear operators on a Hilbert space and bounded sesquilinear forms.
- The weak measure topology and the weak operator topology.
- The WMT-density of $C(X)$ in $B(X)$.
- The separate continuity of the multiplication in $(B(X), \text{WMT})$ and in $(\mathcal{B}(H), \text{WOT})$.
- The Borel functional calculus for a normal operator (definition, uniqueness, a necessary condition for the existence).
- The canonical extension of representations of $C(X)$ to $B(X)$.
- The existence of a Borel functional calculus for a normal operator (Spectral Theorem II).

6. Spectral Theorem III: spectral measures

- Finitely additive spectral measures and associated complex measures. Examples.
- Integration of bounded measurable functions with respect to a finitely additive spectral measure. A 1-1 correspondence between finitely additive spectral measures and representations of $B_\sigma(X)$.
- Regular spectral measures on a compact Hausdorff topological space.
• A 1-1 correspondence between regular spectral measures and WMT-WOT-continuous representations of $B(X)$.
• A characterization of representations of $C(X)$ in terms of regular spectral measures.
• The spectral decomposition of a normal operator (Spectral Theorem III).

*Items marked as “★” are not obligatory for the students who participate in the exam on May 25.*