Functional Analysis 2 (Operator Theory)

Exam 25.05.2021

1. **Topological vector spaces**
   - Topological vector spaces. The topology generated by a family of seminorms. Locally convex spaces and their "polynormability". Examples of locally convex spaces.
   - A characterization of convergence in a locally convex space in terms of seminorms.
   - A criterion for a locally convex space to be Hausdorff.
   - A continuity criterion for a seminorm on a locally convex space.
   - A continuity criterion for a linear operator between locally convex spaces.
   - The domination relation for families of seminorms. Equivalent families of seminorms. Examples.
   - Linear functionals on locally convex spaces (extension from subspaces, separation of points, separation of points and subspaces).
   - Dual pairs of vector spaces. The weak topology of a dual pair. Special cases and basic properties of the weak topology.
   - A characterization of linear functionals that are continuous with respect to the weak topology.
   - A characterization of reflexive Banach spaces in terms of topologies on the dual.
   - Dual operators between dual pairs of vector spaces. An existence criterion for the dual operator in terms of weak continuity.
   - Relations between the continuity, the weak continuity, and the existence of the dual for an operator between locally convex spaces (in particular, between normed spaces).
   - Annihilators. Basic properties of annihilators. The double annihilator theorem (for dual pairs and for locally convex spaces). Corollaries: the weak closure of a vector subspace; a density criterion for a vector subspace; relations between kernels and images of a linear operator and of the dual operator.
   - Equicontinuous families of linear operators. The Banach–Alaoglu–Bourbaki theorem.
2. **Commutative Banach algebras**
   - The maximal spectrum and the character space of a commutative unital algebra.
   - The closedness of maximal ideals of a commutative unital Banach algebra.
   - The 1-1 correspondence between the character space and the maximal spectrum of a commutative unital Banach algebra.
   - An invertibility criterion in terms of characters.
   - The Gelfand topology on the maximal spectrum.
   - The compactness of the maximal spectrum (unital case).
   - The Gelfand transform of a commutative unital Banach algebra.
   - Properties of the Gelfand transform.
   - The maximal spectrum and the Gelfand transform for subalgebras of $C(X)$.

3. **C^*-algebras and continuous functional calculus**

   - Banach $*$-algebras. C^*-algebras. Examples.
   - Selfadjoint elements, unitary elements, normal elements.
   - The main property of the spectral radius of a normal element. Corollaries: the uniqueness of the C^*-norm on a $*$-algebra; the automatic continuity of $*$-homomorphisms.
   - The main property of the spectrum of a selfadjoint element in a C^*-algebra. Corollaries: characters of a C^*-algebra are $*$-characters; the spectral invariance of C^*-subalgebras.
   - The 1st (commutative) Gelfand-Naimark theorem (unital case).
   - The continuous functional calculus in C^*-algebras (definition, a necessary condition for the existence, the existence theorem).
   - The spectral mapping theorem and the superposition property for the functional calculus.

4. **Spectral Theorem I: functional models**

   - $*$-representations and $*$-modules. Cyclic $*$-modules. Examples.
   - The functional model for a cyclic $*$-module over $C(X)$.
   - A correspondence between normal operators and $*$-modules over $C(K)$ ($K \subset \mathbb{C}$).
   - $*$-cyclic operators and their relation to cyclic $*$-modules.
   - The functional model for a $*$-cyclic normal operator.
   - Hilbert direct sums of Hilbert spaces and of $*$-modules.
   - A decomposition of a $*$-module into a Hilbert direct sum of cyclic submodules.
   - The functional model for a $*$-module over $C(X)$.
   - The functional model for a normal operator (Spectral Theorem I).

5. **Spectral Theorem II: Borel functional calculus**

   - A correspondence between bounded linear operators on a Hilbert space and bounded sesquilinear forms*.
   - The weak measure topology and the weak operator topology.
   - The WMT-density of $C(X)$ in $B(X)$.
   - The separate continuity of the multiplication in $(B(X), \mathrm{WMT})$ and in $(\mathscr{B}(H), \mathrm{WOT})$.
   - The Borel functional calculus for a normal operator (definition, uniqueness, a necessary condition for the existence).
   - The canonical extension of representations of $C(X)$ to $B(X)$.
   - The existence of a Borel functional calculus for a normal operator (Spectral Theorem II).

6. **Spectral Theorem III: spectral measures***

   - Finitely additive spectral measures and associated complex measures. Examples.
   - Integration of bounded measurable functions with respect to a finitely additive spectral measure. A 1-1 correspondence between finitely additive spectral measures and representations of $B_{\mathscr{A}}(X)$.
   - Regular spectral measures on a compact Hausdorff topological space.

- A 1-1 correspondence between regular spectral measures and WMT-WOT-continuous representations of $B(X)$.
- A characterization of representations of $C(X)$ in terms of regular spectral measures.
- The spectral decomposition of a normal operator (Spectral Theorem III).

[★]Items marked as "★" are not obligatory for the students who participate in the exam on May 25.