The Riesz–Schauder theory

F. Riesz (1918)
J. Schauder (1930)

$\boxed{\begin{array}{l} 1+K \\ \text{is a Fredholm oper.} \\ \text{of index } 0. \end{array}}$

X = vector space, $T: X \to X$ linear.

<u>Notation</u>. $K^n = \operatorname{Ker} T^n$; $I_n = \operatorname{Im} T^n$ $(n \geq 0)$.

$0 = K_0 \subset K_1 \subset K_2 \subset \ldots$; $X = I_0 \supset I_1 \supset I_2 \supset \ldots$

<u>Def</u> The <u>ascent</u> of T is (подъём)

$$a(T) = \min\{n : K_n = K_{n+i} \ \forall i \geq 0\} \quad \left(a(T) = \infty \text{ if } \nexists \text{ such } n\right)$$

The <u>descent</u> of T is (спуск)

$$d(T) = \min\{n : I_n = I_{n+i} \ \forall i \geq 0\} \quad \left(d(T) = \infty \text{ if } \nexists \text{ such } n\right)$$

<u>Lemma 1</u>. (1) If $K_n = K_{n+1}$, then $K_n = K_{n+i}$ $\forall i \geq 0$
(hence $a(T) < \infty$)

(2) If $I_n = I_{n+1}$, then $I_n = I_{n+i}$ $\forall i \geq 0$
(hence $d(T) < \infty$)

(3) If $a(T) < \infty$ and $d(T) < \infty \Rightarrow a(T) = d(T)$.

Proof. (1) exer.

(2) It suff. to show that $I_{n+1}=I_{n+2}$.

Let $x\in I_{n+1} \Rightarrow x = T^{n+1}y = T(\underbrace{T^n y}_{\in I_n}) = T(T^{n+1}z)$ (for some z)

$= T^{n+2}(z) \in I_{n+2}$.

(3) Let's show that $d(T)\leq a(T)$

Suppose $d(T)>a(T)$, let $n=d(T)-1$

$\Rightarrow I_n \supsetneq I_{n+1} = I_{n+2}$ and $K_n = K_{n+1}$.

Let $x\in I_n \Rightarrow x = T^n y$ $(y\in X)$ $\Rightarrow Tx = \underbrace{T^{n+1}y}_{\in I_{n+1}=I_{n+2}} = T^{n+2}z$

(for some z) $\Rightarrow$

$\Rightarrow y - Tz \in K_{n+1} = K_n \Rightarrow \underset{x}{\underset{\shortparallel}{T^n y}} = T^{n+1}z \in I_{n+1}$

$\Rightarrow I_n = I_{n+1}$, a contr.

$a(T)\leq d(T)$ — exer. $\square$

Lemma 2. (Riesz decomposition)

X = vec. space, $T: X\to X$ lin. oper.

Suppose $a(T)<\infty$ and $d(T)<\infty$.

Let $p = a(T) = d(T)$, $K_p = \operatorname{Ker} T^p$, $I_p = \operatorname{Im} T^p$

Then $X = K_p \oplus I_p$, K_p and I_p are T-invar,

$T|_{K_p}$ is nilpotent, and $T|_{I_p} : I_p \to I_p$ is an isomorph.

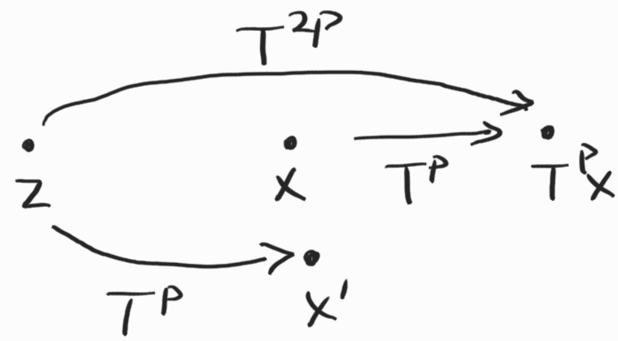
Proof Let $x \in X \Rightarrow T^p x \in I_p = I_{2p} \Rightarrow$

$\Rightarrow T^p x = T^{2p} z$ (for some $z \in X$). Let $x' = T^p z$

$\Rightarrow T^p x' = T^p x \Rightarrow$

$\Rightarrow x - x' \in K_p \Rightarrow$

$\Rightarrow x = \underbrace{x'}_{\in I_p} + \underbrace{(x - x')}_{K_p}$



$\Rightarrow X = K_p + I_p$.

Let $x \in K_p \cap I_p \Rightarrow T^p x = 0$ and $x = T^p y$ $(y \in X)$

$\Rightarrow T^{2p} y = 0$, that is, $y \in K_{2p} = K_p \Rightarrow \underbrace{T^p y}_{x} = 0$

$\Rightarrow K_p \cap I_p = 0 \Rightarrow X = K_p \oplus I_p$.

Clearly, K_p and I_p are T-inv; $T|_{K_p}$ is nilp.

$T(I_p) = I_{p+1} = I_p \Rightarrow T|_{I_p} : I_p \to I_p$ is surj.

$K_p \cap I_p = 0$, $\operatorname{Ker} T = K_1 \subset K_p \Rightarrow (\operatorname{Ker} T) \cap I_p = 0$

$\Rightarrow T|_{I_p}$ is inj. $\Rightarrow T|_{I_p} : I_p \to I_p$ is an isom $\square$

__Thm__ (F. Riesz, 1918)

X = Banach space, $T \in \mathcal{B}(X)$, $T \in 1_X + \mathcal{K}(X)$.

Then:

(1) T is Fredholm, and $\operatorname{ind} T = 0$

(2) $a(T) < \infty$, $d(T) < \infty$.

(3) Let $p = a(T) = d(T)$, $K_p = \operatorname{Ker} T^p$, $I_p = \operatorname{Im} T^p$.

Then K_p and I_p are closed T-inv subspaces, K_p is fin-dim, $X = K_p \oplus I_p$, $T|_{K_p}$ is nilpotent, and $T|_{I_p} : I_p \to I_p$ is a topol isomorphism

__Proof__ $T = 1 - S$, $S \in \mathcal{K}(X)$

__Step 1.__ $\operatorname{Ker} T$ is fin-dim, $\operatorname{Im} T$ is closed.

__Proof__ Let $K = \operatorname{Ker} T \Rightarrow S|_K = 1_K$; $S|_K$ is comp

$\Rightarrow \dim K < \infty$.

Consider $\hat{T} : X/K \to X$, $\hat{T}(x+K) = Tx$.

It suff to show that $\hat{T}$ is topol. injective, that is, $\exists\, c > 0$ s.t. $\forall\, u \in X/K$ $\|\hat{T}u\| \geq c\|u\|$.

In other words, $\|Tx\| \geq c\|x+K\|$ $\forall\, x \in X$.

Assume that there is no such $c > 0$.

Then $\exists$ a seq. (x_n) in X s.t

$\underset{\text{dist}(x_n,K)}{\overset{}{\|x_n+K\|}}=1 \ \forall n$, but $Tx_n \to 0 \ (n\to\infty)$

We may assume that (x_n) is bdd (because $\|x_n+K\|=1$)

We may assume that $\underset{x_n - Tx_n}{Sx_n} \to x \in X$

$\Rightarrow x_n \to x \Rightarrow Tx_n \to Tx$

$\Rightarrow Tx=0$, that is, $x\in K$. This is a contradiction (because $x_n\to x$ and $\text{dist}(x_n,K)=1$) $\square$

<u>Step 2</u>. T is Fredholm.

<u>Proof</u>. We know: $\operatorname{Im}T$ is closed $\Rightarrow \operatorname{Coker}T$ is a Ban. space. $\quad T^*=1-S^*\in 1+\mathcal{K}(X^*)$

$\Rightarrow \operatorname{Ker}T^*$ is fin-dim.

We know: $(\operatorname{Coker}T)^*\cong \operatorname{Ker}T^* \Rightarrow \operatorname{Coker}T$ is fin-dim

$\Rightarrow T$ is Fredholm. $\square$

<u>Step 3</u>. $a(T)<\infty$.

<u>Proof</u>. Assume $a(T)=\infty$. Hence $K_n \subsetneq K_{n+1} \ \forall n$

$\forall n$ let $x_n\in K_{n+1}$ be a $\frac{1}{2}$-$\perp$ to K_n

(that is, $\|x_n\|=1$ and $\text{dist}(x_n,K_n)\geq \frac{1}{2}$).

$\forall n,m\in\mathbb{N}$, $n\geq m+1$, we have

$\|Sx_n - Sx_m\| = \|(1-T)x_n-(1-T)x_m\| =$

$$= \| x_n - \underbrace{Tx_n}_{\in K_n} - \underbrace{x_m + Tx_m}_{K_{m+1} \subset K_n} \| \geq \frac{1}{2},$$ which is a contradiction with the compactness of S. $\square$

<u>Step 4</u> $d(T) < \infty$.

<u>Proof</u>. We know that $a(T^*) < \infty$, that is, $\exists\, n \in \mathbb{N}$ s.t. $\operatorname{Ker}(T^*)^n = \operatorname{Ker}(T^*)^{n+1}$.

$$\Rightarrow \quad {}^{\perp}\operatorname{Ker}(T^n)^* = {}^{\perp}\operatorname{Ker}(T^{n+1})^*$$

$$\begin{matrix} {}^{\perp}\operatorname{Ker}(T^n)^* & = & {}^{\perp}\operatorname{Ker}(T^{n+1})^* \\ \| & & \| \\ \overline{\operatorname{Im} T^n} & & \overline{\operatorname{Im} T^{n+1}} \\ \| & & \| \\ \operatorname{Im} T^n & & \operatorname{Im} T^{n+1} \end{matrix}$$

$\Rightarrow d(T) < \infty$ by L1 $\square$

<u>Step 5</u>. T^p is Fredholm $\Rightarrow I_p$ is closed, K_p is fin-dim.

L2 $\Rightarrow$ (3) (because $T|_{I_p}$ is a top. isom by Banach's Inverse Mapping Thm)

$$\operatorname{ind}(T) = \underbrace{\operatorname{ind}(T|_{K_p})}_{=0} + \underbrace{\operatorname{ind}(T|_{I_p})}_{=0} = 0 \qquad \square\square$$

<u>Cor. 1</u> (<u>the abstract Fredholm alternative</u>)

$T \in 1 + \mathcal{K}(X)$ is injective $\Leftrightarrow T$ is surj
$\Leftrightarrow T$ is bijective.

<u>Cor. 2</u> (<u>J. Schauder 1930</u>)
(<u>abstract Fredholm's theorems</u>)

$T \in 1 + \mathcal{K}(X)$. Then

(1) $\dim \operatorname{Ker} T = \dim \operatorname{Ker} T^* < \infty$.

(2) $\operatorname{Im} T = {}^{\perp}\operatorname{Ker} T^*$ (3) $\operatorname{Im} T^* = (\operatorname{Ker} T)^{\perp}$.

<u>The spectrum of a compact operator</u>

<u>Recall</u>: X = Ban. space over $\mathbb{C}$, $T \in \mathcal{B}(X)$.

The <u>spectrum</u> of T is

$$\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda 1_X \text{ is not invertible}\}$$

The <u>point spectrum</u> of T is

$$\sigma_p(T) = \{\lambda \in \mathbb{C} : \operatorname{Ker}(T - \lambda 1_X) \neq 0\} \subset \sigma(T)$$

<u>Thm.</u> X = Ban. space, $T \in \mathcal{K}(X)$. Then:

(1) $\lambda \in \sigma(T) \setminus \{0\} \Rightarrow \lambda \in \sigma_p(T)$, λ has finite multiplicity (that is, $\dim \operatorname{Ker}(T - \lambda 1) < \infty$), and λ is an isolated point of $\sigma(T)$

(2) $\sigma(T)$ is at most countable

<u>Proof</u> (1) $T-\lambda 1 = -\lambda(1-\lambda^{-1}T) \Longrightarrow$

$\Rightarrow$ the Riesz-Schauder theory applies to $T-\lambda 1$.

The Fredholm alt. $\Rightarrow$ $T-\lambda 1$ is not inj., that is, $\lambda \in \sigma_p(T)$

$T-\lambda 1$ is Fredholm $\Rightarrow$ $\operatorname{Ker}(T-\lambda 1)$ is fin-dim

Let $p = a(T-\lambda 1) = d(T-\lambda 1)$, $K_p = \operatorname{Ker}(T-\lambda 1)^p$, $I_p = \operatorname{Im}(T-\lambda 1)^p$. $\Rightarrow$ K_p and I_p are closed and T-invariant, and $X = K_p \oplus I_p$ $\Longrightarrow$

$\Longrightarrow_{(*)}$ $\boxed{\sigma(T) = \sigma(T|_{K_p}) \cup \sigma(T|_{I_p})}$; $K_p \neq 0$..

$(T-\lambda 1)|_{K_p}$ is nilp. $\Rightarrow$ $\sigma((T-\lambda 1)|_{K_p}) = \{0\}$

$\Rightarrow \sigma(T|_{K_p}) = \{\lambda\}$

$(T-\lambda 1)|_{I_p} : I_p \to I_p$ is a top. isomorphism, that is, $0 \notin \sigma((T-\lambda 1)|_{I_p})$ $\Rightarrow$ $\lambda \notin \sigma(T|_{I_p})$.

Now $(*)$ becomes $\sigma(T) = \{\lambda\} \sqcup \sigma(T|_{I_p})$

But $\sigma(T|_{I_p})$ is compact $\Rightarrow$ closed in $\sigma(T)$

$\Rightarrow \lambda$ is an isolated point of $\sigma(T)$

(2) $\forall n \in \mathbb{N}$ let $K_n = \sigma(T) \cap \{\lambda \in \mathbb{C} : |\lambda| \geq \frac{1}{n}\}$

K_n is compact, and all points of K_n are isolated $\Rightarrow$ K_n is finite $\Rightarrow$

$\sigma(T) \setminus \{0\} = \bigcup_{n=1}^{\infty} K_n$ is at most countable. $\square$