# Spectra

This exercise sheet is meant to be discussed at the blackboard during a "traditional" seminar, and is also intended for self-study (that is, is plays no role in the grading). Nevertheless, it is strongly recommended to look at it before the exam.

**5.1.** Show that for each subset $S \subset \mathbb{C}$ there exist a unital algebra $A$ and $a \in A$ such that $\sigma_A(a) = S$. (Do not forget about $S = \varnothing$.)

**5.2.** Show that for each nonempty compact subset $K \subset \mathbb{C}$ there exists a bounded linear operator $T$ on a Banach space such that $\sigma(T) = K$.

**5.3.** Prove that the spectrum of a bijective isometry on a Banach space is contained in $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.

**5.4.** Find the point spectrum, the continuous spectrum, and the residual spectrum of the diagonal operator on $\ell^\infty$.

**5.5.** Let $(X, \mu)$ be a measure space, and let $f\colon X \to \mathbb{C}$ be a measurable function. Recall (see the lectures) that $\lambda \in \mathbb{C}$ is an *essential value* of $f$ if for each neighborhood $U \ni \lambda$ we have $\mu(f^{-1}(U)) > 0$. The set of all essential values of $f$ is called the *essential range* of $f$. Also recall (see the lectures) that, if $f$ is essentially bounded, then the spectrum $\sigma_{L^\infty(X,\mu)}(f)$ is equal to the essential range of $f$.
**(a)** Show that $f(X)$ is not necessarily contained in the essential range of $f$.
**(b)** Show that the essential range of $f$ is not necessarily contained in $f(X)$.
**(c)** Show that, if $X = [a, b]$ or $X = \mathbb{T}$ with the Lebesgue measure, and if $f$ is continuous, then the essential range of $f$ is equal to $f(X)$.

**5.6.** Let $(X, \mu)$ be a $\sigma$-finite measure space, $f$ be an essentially bounded measurable function on $X$, and $M_f$ be the multiplication operator on $L^p(X, \mu)$ acting by the rule $g \mapsto fg$ (where $1 \le p \le \infty$). Find the point spectrum, the continuous spectrum, and the residual spectrum of $M_f$. Pay special attention to the case of $M_t\colon L^2[0, 1] \to L^2[0, 1]$, $(M_t g)(t) = tg(t)$.

**5.7.** Find the point spectrum, the continuous spectrum, and the residual spectrum of the operator $T\colon L^2[-\pi, \pi] \to L^2[-\pi, \pi]$ acting by the rule

$$(Tf)(t) = \int_{-\pi}^{\pi} \sin^2(t - s) f(s)\, ds.$$

(*Hint*: replace $T$ by a unitary equivalent operator on $\ell^2(\mathbb{Z})$.)

**5.8.** Find the point spectrum, the continuous spectrum, and the residual spectrum of the left and right shift operators on **(a)** $c_0$; **(b)** $\ell^1$; **(c)-B** $\ell^\infty$.

**5.9.** Given $\zeta \in \mathbb{T}$, define the shift operator $T_\zeta\colon L^2(\mathbb{T}) \to L^2(\mathbb{T})$ by $(T_\zeta f)(z) = f(\zeta^{-1} z)$. Find the point spectrum, the continuous spectrum, and the residual spectrum of $T_\zeta$.

**5.10** (*the Volterra operator*). Let $I = [a, b]$, let $H = L^2(I)$, and let $K \in L^2(I \times I)$. The *Volterra operator* $V_K\colon L^2(I) \to L^2(I)$ is given by

$$(V_K f)(x) = \int_a^x K(x, y) f(y)\, dy$$

**(a)** Prove that $V_K$ is quasinilpotent whenever $K$ is bounded.
**(b)-B** Prove that $V_K$ is quasinilpotent for each $K \in L^2(I \times I)$.