

Duality for normed spaces

4.1. Recall from the lectures that if $1 < p, q < +\infty$ and $1/p + 1/q = 1$, then there exists an isometric isomorphism $\ell^q \xrightarrow{\cong} (\ell^p)^*$. By using a similar argument, construct isometric isomorphisms **(a)** $\ell^\infty \xrightarrow{\cong} (\ell^1)^*$; **(b)** $\ell^1 \xrightarrow{\cong} (c_0)^*$. Does this approach give an isometric isomorphism $\ell^1 \cong (\ell^\infty)^*$?

4.2. Describe explicitly the duals of the following operators:

- (a)** the diagonal operator on ℓ^p (where $1 \leq p < \infty$) or on c_0 ;
- (b)** the right shift operator on ℓ^p (where $1 \leq p < \infty$) or on c_0 ;
- (c)** the operator of “taking the primitive” on $L^2[0, 1]$ (see Exercise 2.6);
- (d)** the Hilbert-Schmidt integral operator on $L^2(X, \mu)$ (see Exercise 2.8).

4.3-B. Prove that c_0 is not isomorphic to the dual of a normed space.

4.4. Let X be a normed space.

- (a)** Prove that if X^* is separable, then so is X .
- (b)** Is the converse true?
- (c)** Prove that there is no topological isomorphism between $(\ell^\infty)^*$ and ℓ^1 .

4.5. Let X be a normed space, and let $i_X: X \rightarrow X^{**}$ be the canonical embedding. Prove that for each operator $T \in \mathcal{B}(X, Y)$ the following diagram commutes.

$$\begin{array}{ccc} X^{**} & \xrightarrow{T^{**}} & Y^{**} \\ i_X \uparrow & & \uparrow i_Y \\ X & \xrightarrow{T} & Y \end{array}$$

4.6. Prove that the composition of the canonical embedding $c_0 \rightarrow (c_0)^{**}$ and the standard isomorphism $(c_0)^{**} \cong \ell^\infty$ is the inclusion of c_0 into ℓ^∞ . Deduce that c_0 is not reflexive.

4.7. Prove that **(a)** a Hilbert space is reflexive; **(b)** ℓ^1 is not reflexive; **(c)** $L^1[a, b]$ is not reflexive; **(d)** $C[a, b]$ is not reflexive.

4.8. Let X be a normed space, and let $i_X: X \rightarrow X^{**}$ be the canonical embedding. Find a relation between the operators $i_{X^*}: X^* \rightarrow X^{***}$ and $i_X^*: X^{***} \rightarrow X^*$.

- 4.9. (a)** Prove that a Banach space X is reflexive $\iff X^*$ is reflexive.
- (b)** Deduce that $\ell^1, \ell^\infty, L^\infty[a, b]$ are not reflexive.

4.10. Let X and Y be Banach spaces, and let $S \in \mathcal{B}(Y^*, X^*)$. Do we always have $S = T^*$ for some $T \in \mathcal{B}(X, Y)$?

4.11. Identify $(\ell^1)^*$ with ℓ^∞ (see Exercise 4.1), and consider c_0 as a subspace of $(\ell^1)^*$. Find ${}^\perp c_0$ and $({}^\perp c_0)^\perp$.

4.12. Let X be a nonreflexive Banach space. Prove that there exists a closed vector subspace $N \subseteq X^*$ such that $N \neq ({}^\perp N)^\perp$.

4.13. Give an example of an injective operator $T \in \mathcal{B}(X, Y)$ between Banach spaces X and Y such that $\text{Im } \overline{T^*}$ is not dense in X^* . (*Hint:* X must be nonreflexive, see the lectures.) As a corollary, the equality $\overline{\text{Im}(T^*)} = (\text{Ker } T)^\perp$ can fail in the nonreflexive case.

4.14. Let X be a normed space, and let $X_0 \subset X$ be a closed vector subspace. Construct isometric isomorphisms $(X/X_0)^* \cong X_0^\perp$ and $X_0^* \cong X^*/X_0^\perp$. (*Hint:* use the universal property of quotients.)

The three basic principles of Functional Analysis

(Hahn-Banach, Banach-Steinhaus, Open Mapping Theorem)

4.15. Let $X = \mathbb{R}^2$ equipped with the norm $\|\cdot\|_p$, and let $X_0 = \{(x, 0) : x \in \mathbb{R}\} \subset X$. Define a linear functional $f_0: X_0 \rightarrow \mathbb{R}$ by $f_0(x, 0) = x$. We clearly have $\|f_0\| = 1$. Describe all “Hahn-Banach extensions” of f_0 , i.e., all linear functionals $f: X \rightarrow \mathbb{R}$ such that $f|_{X_0} = f_0$ and $\|f\| = 1$. (Consider all possible $p \in [1, +\infty]$.)

4.16. Give an example of a normed space X and a pointwise bounded sequence (f_n) in X^* such that (f_n) is not norm bounded.

4.17. Let X, Y, Z be normed spaces.

(a) Prove that a bilinear operator $T: X \times Y \rightarrow Z$ is continuous if and only if there exists $C \geq 0$ such that $\|T(x, y)\| \leq C\|x\|\|y\|$ for all $x \in X, y \in Y$.

(b) Assume that either X or Y is complete. Prove that each separately continuous bilinear operator $X \times Y \rightarrow Z$ is continuous. (The separate continuity means that for each $x_0 \in X, y_0 \in Y$ the maps $Y \rightarrow Z, y \mapsto T(x_0, y)$, and $X \rightarrow Z, x \mapsto T(x, y_0)$, are continuous.) *Hint:* use the Uniform Boundedness Principle.

(c) Does (b) hold without the completeness assumption?

4.18-B. Let G be a compact topological group, and let π be a representation of G on a Banach space X . Suppose that π is continuous in the sense that the map $G \times X \rightarrow X, (g, x) \mapsto \pi(g)x$, is continuous. Prove that there exists an equivalent norm $\|\cdot\|_\pi$ on X such that all the operators $\pi(g)$ are isometric with respect to $\|\cdot\|_\pi$. (*Warning:* this has nothing to do with the Haar measure!).

4.19. (a) Deduce the Open Mapping Theorem from the Inverse Mapping Theorem.

(b) Deduce the Inverse Mapping Theorem from the Closed Graph Theorem.

(c)-**B** Deduce the Uniform Boundedness Principle from the Closed Graph Theorem.

4.20. (a) Give an example of a Banach space X , a normed space Y , and a bijective operator $T \in \mathcal{B}(X, Y)$ such that T^{-1} is unbounded.

(b)-**B** Give an example of a normed space X , a Banach space Y , and a bijective operator $T \in \mathcal{B}(X, Y)$ such that T^{-1} is unbounded.

4.21. Let $\|\cdot\|$ be a norm on $L^1(\mathbb{R})$ such that $(L^1(\mathbb{R}), \|\cdot\|)$ is complete and such that the convergence $f_n \rightarrow f$ with respect to $\|\cdot\|$ implies that $\int_{-\infty}^t f_n(s) ds \rightarrow \int_{-\infty}^t f(s) ds$ for all $t \in \mathbb{R}$. Prove that $\|\cdot\|$ is equivalent to the usual L^1 -norm.