Hilbert spaces

Exercises marked by "**-B**" are optional. If you solve such exercises, you will earn bonus points.

3.1. Show that the norm on the spaces $(\mathbb{C}^n, \|\cdot\|_p)$, ℓ^p, $(C[a,b], \|\cdot\|_p)$, $L^p(X,\mu)$ (where (X,μ) is a measure space containing at least two disjoint measurable sets of positive measure) is not generated by an inner product (unless $n=1$, $p=2$).

3.2. Generalize the parallelogram rule to n vectors.

3.3. Show that the norm on the spaces ℓ^p, $(C[a,b], \|\cdot\|_p)$, $L^p(X,\mu)$ (where (X,μ) is a measure space containing infinitely many disjoint measurable sets of positive measure) is not equivalent to a norm generated by an inner product (unless $p=2$).

3.4. Consider the vector space $H = C[-1,1]$ with the inner product $\langle f \,|\, g\rangle = \int_{-1}^1 f(t)\overline{g(t)}\,dt$. Let

$$H_0 = \left\{ f \in H : \int_{-1}^0 f(t)\,dt = \int_0^1 f(t)\,dt \right\}.$$

(a) Prove that H_0 is a closed vector subspace of H.
(b) Does the equality $H = H_0 \oplus H_0^\perp$ hold?

3.5. Prove that every incomplete inner product space H has a closed vector subspace H_0 such that $H_0 \oplus H_0^\perp \neq H$.

3.6. Let $C_c^\infty(a,b)$ be the space of smooth compactly supported functions on the interval (a,b). Prove that for each $p \in [1,\infty)$ $C_c^\infty(a,b)$ is dense in $L^p[a,b]$.

Definition 3.1. Let $f \in L^2[a,b]$. A function $f' \in L^2[a,b]$ is a *weak derivative* of f if

$$\int_a^b f'\varphi\,dt = -\int_a^b f\varphi' dt$$

for all $\varphi \in C_c^\infty(a,b)$.

3.7. Prove that if $f \in L^2[a,b]$ has a weak derivative f', then f' is unique (as an element of $L^2[a,b]$).

3.8 (*the Sobolev space*)**.** Let $W^{1,2}(a,b)$ denote the space of all $f \in L^2[a,b]$ that have a weak derivative $f' \in L^2[a,b]$. Prove that $W^{1,2}(a,b)$ is a Hilbert space with respect to the inner product

$$\langle f \,|\, g\rangle = \int_a^b (f\bar{g} + f'\bar{g}')\,dt.$$

3.9 (*the Hardy space*)**.** Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and let H^2 denote the space of holomorphic functions $f\colon \mathbb{D} \to \mathbb{C}$ satisfying the following condition:

$$\|f\| = \sup_{0<r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\varphi})|^2\,d\varphi \right)^{1/2} < \infty.$$

Show that the map $f \mapsto (c_n(f))_{n\geqslant 0}$ (where $c_n(f)$ is the nth Taylor coefficient of f at 0) is an isometric isomorphism of $(H^2, \|\cdot\|)$ onto $\ell^2(\mathbb{Z}_{\geqslant 0})$. Hence H^2 is a Hilbert space.

3.10-B (*the Bergman space*)**.** Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and let $L_a^2(\mathbb{D})$ denote the space of holomorphic functions $f\colon \mathbb{D} \to \mathbb{C}$ satisfying the following condition:

$$\|f\| = \left(\int_{\mathbb{D}} |f(x+iy)|^2\,dx\,dy \right)^{1/2} < \infty.$$

Show that $L_a^2(\mathbb{D})$ is a closed vector subspace of $L^2(\mathbb{D})$. Hence $L_a^2(\mathbb{D})$ is a Hilbert space.