Introduction to Functional Analysis

Exam 24.12.2020

1. The Hahn–Banach theorem. Corollaries: extension of bounded linear functionals, a "dual" formula for the norm, linear functionals separate the points, linear functionals separate the points and vector subspaces.

2. Dual spaces and dual operators. Basic properties of dual operators.

3. The Riesz Representation Theorem for linear functionals on a Hilbert space.

4. The dual of ℓ^p $(1 < p < \infty)$.

5. Similar and isometrically equivalent operators. The duals of diagonal and shift operators on ℓ^p $(1 \leqslant p < \infty)$.

6. Complex measures. The variation of a complex measure. The integral of a bounded measurable function w.r.t. a complex measure. Complex Radon measures on compact topological spaces. The dual of $C(X)$.

7. The canonical embedding of a normed space into the bidual. The "naturality" of the canonical embedding. Reflexive Banach spaces. Examples.

8. Annihilators and preannihilators, their basic properties. The double annihilator theorem. The density criterion for a vector subspace of a normed space.

9. Relations between kernels and images of operators and of their duals. Duality between injective operators and operators with dense image. Duality for topological (and isometric) isomorphisms.

10. Barrels in normed spaces. The barrel lemma for Banach spaces.

11. The Uniform Boundedness Principle (the Banach–Steinhaus theorem). Corollaries: a boundedness criterion for a subset of a normed space, pointwise convergence of operators.

12. The Open Mapping Theorem, the Inverse Mapping Theorem, the Closed Graph Theorem.

13. The spectrum of an algebra element. Examples: the spectra of elements of $\mathbb{C}^X$, $\ell^\infty(X)$, $L^\infty(X, \mu)$.

14. The behavior of the spectrum under homomorphisms. Spectrally invariant subalgebras. Examples.

15. The polynomial spectral mapping theorem. The spectrum of the inverse element.

16. Banach algebras. Examples.

17. Properties of the group of invertibles in a Banach algebra. The automatic continuity of characters (i.e., of $\mathbb{C}$-valued homomorphisms).

18. The compactness of the spectrum of a Banach algebra element.

19. The resolvent function and its properties. The nonemptiness of the spectrum of a Banach algebra element. The Gelfand–Mazur theorem.

20. The spectral radius of a Banach algebra element. Examples. The Beurling–Gelfand formula. Corollaries: a characterization of quasinilpotents, the spectral radius w.r.t. a closed subalgebra.

21. The point spectrum, the continuous spectrum, and the residual spectrum of a bounded linear operator.

22. Calculating the parts of the spectrum for diagonal operators, for multiplication operators, and for the bilateral shift on $\ell^2(\mathbb{Z})$.

23. The spectrum of the dual operator. Inclusions between the parts of the spectrum of an operator and of the dual operator.

24. Calculating the parts of the spectrum for the shift operators on ℓ^p, $1 < p < \infty$.