Introduction to Functional Analysis

Midterm exam 05.11.2020

1. Normed spaces. Examples: finite-dimensional normed spaces, sequence spaces, function spaces.
2. Bounded linear operators. Characterizations of bounded linear operators. Boundedness = continuity. The norm of a bounded linear operator.
3. Examples of bounded linear operators: multiplication operators, shift operators, integral operators.
4. Topologically injective operators, isometries, topological and isometric isomorphisms. Topologically injective = bounded below.
5. The domination relation and the equivalence of norms on a vector space. Any two norms on a finite-dimensional vector space are equivalent.
6. Banach spaces. Completeness is a linear topological property. Examples (and counterexamples) of Banach spaces.
7. The completeness of $\mathscr{B}(X, Y)$.
8. The "extension by continuity" theorem.
9. Completions of normed spaces: definition and existence.
10. The universal property of completions.
11. The "uniqueness" of completions.
12. The functoriality of completions.
13. Open operators and coisometries. Characterizations of open operators.
14. Quotients of normed spaces.
15. The universal property of quotients. Corollaries (in particular, conditions under which $X/\operatorname{Ker} T \cong \operatorname{Im} T$ topologically or isometrically).
16. The completeness of quotients.
17. Inner product spaces. The Cauchy–Bunyakowski–Schwarz inequality.
18. The norm generated by an inner product. Hilbert spaces. Examples.
19. Orthogonal complements and their basic properties.
20. Orthogonal projections. A characterization of orthogonal projections in terms of nearest points.
21. The existence of orthogonal projections onto a closed subspace of a Hilbert space. The orthogonal complement theorem. The double orthogonal complement.
22. Orthonormal families in inner product spaces. Examples. The cardinality of orthonormal families in a separable space.

23. Fourier coefficients. An explicit formula for the projection onto a finite-dimensional subspace in terms of Fourier coefficients. Bessel's inequality.

24. Fourier series and their elementary properties (uniqueness, Parseval's identity).

25. Orthonormal bases, total orthonormal families, maximal orthonormal families. Relations between these notions (for general inner product spaces and for Hilbert spaces).

26. Orthogonalization. The existence of an orthonormal basis in a separable inner product space. Examples of orthonormal bases.

27. Classification of separable Hilbert spaces. The Riesz–Fischer theorem.