Compact quantum groups

(EXERCISES FOR LECTURES 14–16)

6.1. Let A be a commutative unital C^*-bialgebra, and let $G = \operatorname{Max} A$. Define a semigroup structure on G, and show that the Gelfand transform $\Gamma_A \colon A \to C(G)$ is a C^*-bialgebra morphism.

6.2. Given a discrete group G, let λ_G denote the left regular representation of G on $\ell^2(G)$ given by $(\lambda_G(x)f)(y) = f(x^{-1}y)$ $(x, y \in G)$.
(a) Show that λ_G is the restriction to G of the homomorphism $\ell^1(G) \to \mathscr{B}(\ell^2(G))$ defined in Exercise 2.7.
(b) Let G and H be discrete groups. Show that there exists an isometric $*$-isomorphism

$$C_r^*(G) \otimes_* C_r^*(H) \xrightarrow{\sim} C_r^*(G \times H), \qquad \lambda_G(x) \otimes \lambda_H(y) \mapsto \lambda_{G\times H}(x, y).$$

(c) Show that there exists a unital $*$-homomorphism

$$\Delta \colon C_r^*(G) \to C_r^*(G) \otimes_* C_r^*(G), \qquad \lambda_G(x) \mapsto \lambda_G(x) \otimes \lambda_G(x).$$

(d) Show that $(C_r^*(G), \Delta)$ is a compact quantum group.
Hint to (c): consider the operator W on $\ell^2(G \times G)$ given by $(Wf)(x, y) = f(x, x^{-1}y)$, and calculate W^*TW, where $T \in \operatorname{span}\{\lambda(x) \otimes \lambda(x) : x \in G\}$.

6.3. Let G be discrete group, and let $C^*(G) = C^*(\mathbb{C}G)$ be the (full) group C^*-algebra of G (see the lectures). Show that there exists a unital $*$-homomorphism

$$\Delta \colon C^*(G) \to C^*(G) \otimes_* C^*(G), \qquad U(x) \mapsto U(x) \otimes U(x),$$

where $U(x)$ is the canonical image of $x \in G$ in $C^*(G)$. Prove that $(C^*(G), \Delta)$ is a compact quantum group.

6.4. Suppose that G is finitely generated. Show that $C^*(G)$ and $C_r^*(G)$ (see Exercises 6.2 and 6.3) are compact matrix quantum groups.

6.5. Let $q \in [-1, 1]$, $q \neq 0$. Recall (see the lectures) that

$$C_q(\mathrm{SU}(2)) = C^*\left(\alpha, \gamma \,\middle|\, \left(\begin{smallmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{smallmatrix}\right) \text{ is unitary}\right).$$

(a) Write explicitly the defining relations between $\alpha, \gamma, \alpha^*, \gamma^*$.
(b) Let $u = \left(\begin{smallmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{smallmatrix}\right) \in M_2(C_q(\mathrm{SU}(2)))$. Show that there exists a unique comultiplication $\Delta \colon C_q(\mathrm{SU}(2)) \to C_q(\mathrm{SU}(2)) \otimes_* C_q(\mathrm{SU}(2))$ such that $(C_q(\mathrm{SU}(2)), \Delta, u)$ is a compact matrix quantum group.

Definition 6.1. Let $q \in \mathbb{C}\backslash\{0\}$. The algebra of *regular functions on the quantum* $\mathrm{SL}(2)$ is the unital algebra $\mathscr{O}_q(\mathrm{SL}(2))$ generated by four elements a, b, c, d with relations

$$ab = qba, \quad ac = qca, \quad bd = qdb, \quad cd = qdc, \quad bc = cb,$$
$$ad - da = (q - q^{-1})bc, \quad ad - qbc = 1.$$

6.6. Suppose that $q \in \mathbb{R} \setminus \{0\}$. Show that
(a) There exists an involution on $\mathscr{O}_q(\mathrm{SL}(2))$ uniquely determined by $a^* = d$, $b^* = -qc$ (cf. Exercise 6.6 (b)).
(b) If $|q| \leqslant 1$, then there exists a unital $*$-homomorphism $r\colon \mathscr{O}_q(\mathrm{SL}(2)) \to C_q(\mathrm{SU}(2))$ uniquely determined by $a \mapsto \alpha$, $c \mapsto \gamma$.
(c) $(C_q(\mathrm{SU}(2)), r)$ is the C^*-envelope of $\mathscr{O}_q(\mathrm{SL}(2))$ (cf. Exercise 5.5 (d)).

6.7. Let H be a Hilbert space with an orthonormal basis $\{e_{r,s} : r \in \mathbb{Z}_{\geqslant 0},\ s \in \mathbb{Z}\}$, and let $q \in [-1,1] \setminus \{0\}$.
(a) Show that there exists a $*$-representation π of $\mathscr{O}_q(\mathrm{SL}(2))$ on H uniquely determined by

$$\pi(a)e_{r,s} = \sqrt{1-q^{2r}}\, e_{r-1,s}, \quad \pi(c)e_{r,s} = q^r e_{r,s+1}$$

(here we let $e_{r,s} = 0$ for $r < 0$).
(b) Given $i \in \mathbb{Z}$ and $j, k \in \mathbb{Z}_{\geqslant 0}$, let

$$a_{ijk} = \begin{cases} \alpha^i (\gamma^*)^j \gamma^k, & i \geqslant 0, \\ (\alpha^*)^{-i} (\gamma^*)^j \gamma^k, & i < 0. \end{cases}$$

Show that the set $\{a_{ijk} : i \in \mathbb{Z},\ j,k \in \mathbb{Z}_{\geqslant 0}\}$ is linearly independent in $C_q(\mathrm{SU}(2))$.
(c) Deduce from (b) that the canonical map $\mathscr{O}_q(\mathrm{SL}(2)) \to C_q(\mathrm{SU}(2))$ (see Exercise 6.6 (b)) is injective.

Hint to (b). Extend π to a $*$-representation of $C_q(\mathrm{SU}(2))$ (see Exercise 6.6 (c)). Calculate explicitly $\pi(a_{ijk})e_{r,0}$. Then take a nontrivial linear combination x of the a_{ijk}'s, and look at the decay rate of the Fourier coefficients of $\pi(x)e_{r,0}$ as $r \to \infty$.

Given a unital algebra A, define a linear map $\eta\colon \mathbb{C} \to A$ by $1_{\mathbb{C}} \to 1_A$, and let $\mu\colon A \otimes A \to A$ denote the multiplication in A.

Definition 6.2. A *Hopf algebra* is a bialgebra (A, Δ) equipped with an algebra homomorphism $\varepsilon\colon A \to \mathbb{C}$ (a *counit*) and a linear map $S\colon A \to A$ (an *antipode*) such that $(\varepsilon \otimes \mathbf{1}_A)\Delta = (\mathbf{1}_A \otimes \varepsilon)\Delta = \mathbf{1}_A$ and $\mu(S \otimes \mathbf{1}_A)\Delta = \mu(\mathbf{1}_A \otimes S)\Delta = \eta\varepsilon$.

6.8. **(a)** Show that $\mathscr{O}(\mathrm{SL}(2))$ becomes a Hopf algebra if we define ε and S by $\varepsilon(f) = f(e)$ and $(Sf)(x) = f(x^{-1})$ ($f \in \mathscr{O}(\mathrm{SL}(2))$, $x \in \mathrm{SL}(2)$).
(b) Show that ε and S are uniquely determined by $\varepsilon(a) = \varepsilon(d) = 1$, $\varepsilon(b) = \varepsilon(c) = 0$, $S(a) = d$, $S(d) = a$, $S(b) = -b$, $S(c) = -c$ (for notation, see Exercise 5.5).

6.9. **(a)** Let $q \in \mathbb{C} \setminus \{0\}$. Show that $\mathscr{O}_q(\mathrm{SL}(2))$ is a Hopf algebra with ε and S uniquely determined by $\varepsilon(a) = \varepsilon(d) = 1$, $\varepsilon(b) = \varepsilon(c) = 0$, $S(a) = d$, $S(d) = a$, $S(b) = -q^{-1}b$, $S(c) = -qc$.
(b) Let $q \in [-1,1] \setminus \{0\}$. Identify $\mathscr{O}_q(\mathrm{SL}(2))$ with the dense $*$-subalgebra of $C_q(\mathrm{SU}(2))$ generated by α and γ (see Exercise 6.7 (c)). Show that the antipode S of $\mathscr{O}_q(\mathrm{SL}(2))$ is unbounded (and hence it has no reasonable extension to $C_q(\mathrm{SU}(2))$.

6.10. Let G be a discrete group. Show that the Haar state on $C^*_r(G)$ (see Exercise 6.2) is given by $h(a) = \langle a\delta_e \mid \delta_e \rangle$, where $\delta_e \in \ell^2(G)$ is 1 at e, 0 elsewhere.