

Approximate identities. Positive functionals. GNS construction

(EXERCISES FOR LECTURES 7–9)

3.1. Let X be a locally compact Hausdorff topological space.

(a) Construct an approximate identity in $C_0(X)$.

(b) Show that $C_0(X)$ has a sequential approximate identity if and only if X is σ -compact.

3.2. Let H be a Hilbert space.

(a) Construct an approximate identity in $\mathcal{K}(H)$.

(b) Show that $\mathcal{K}(H)$ has a sequential approximate identity if and only if H is separable.

3.3. Prove that every separable C^* -algebra has a sequential approximate identity.

Recall (see the lectures) that each **closed** two-sided ideal of a C^* -algebra is selfadjoint.

3.4. Construct a nonselfadjoint ideal in $C(\overline{\mathbb{D}})$ (where $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$).

3.5. Show that each two-sided ideal in $\mathcal{B}(H)$ is selfadjoint.

3.6. Let H be an infinite-dimensional separable Hilbert space, $\{e_n : n \geq 0\}$ be an orthonormal basis of H , and S be the right shift operator on H (i.e., a bounded linear operator uniquely determined by $S(e_n) = e_{n+1}$ for all n). Recall that the *Toeplitz algebra* is the C^* -subalgebra \mathcal{T} of $\mathcal{B}(H)$ generated by S .

(a) Identify H with the Hardy space H^2 by taking each e_n to the function $e_n(z) = z^n$ on \mathbb{T} . Prove that \mathcal{T} is generated by the set of all Toeplitz operators on H^2 with continuous symbols.

(b) Prove that $\mathcal{K}(H) \subset \mathcal{T}$, and that $\mathcal{K}(H)$ is the smallest closed 2-sided ideal of \mathcal{T} containing all commutators $[a, b]$ ($a, b \in \mathcal{T}$).

(c) Construct an isometric $*$ -isomorphism $\mathcal{T}/\mathcal{K}(H) \cong C(\mathbb{T})$.

(d) Show that the extension $0 \rightarrow \mathcal{K}(H) \rightarrow \mathcal{T} \xrightarrow{p} C(\mathbb{T}) \rightarrow 0$ does not split (i.e., there is no $*$ -homomorphism $j: C(\mathbb{T}) \rightarrow \mathcal{T}$ satisfying $pj = \mathbf{1}_{C(\mathbb{T})}$). (*Hint*: use the Fredholm index.)

3.7. Prove that a linear functional $f: M_n \rightarrow \mathbb{C}$ is a state if and only if there exists a positive $S \in M_n$ with $\text{Tr}(S) = 1$ such that $f(T) = \text{Tr}(ST)$ ($T \in M_n$).

3.8. Let A be a C^* -algebra, and let $\pi: A \rightarrow \mathcal{B}(H)$ be a nondegenerate $*$ -representation of A on a Hilbert space H . Given $h \in H$ with $\|h\| = 1$, define $f_h: A \rightarrow \mathbb{C}$ by $f_h(a) = \langle \pi(a)h | h \rangle$. Show that f_h is a state.

3.9. Let G be a discrete group. Define $f: \ell^1(G) \rightarrow \mathbb{C}$ by $f(a) = a(e)$ (where e is the identity of G). Prove that f is a state, and that f uniquely extends to a state on $C_r^*(G)$ (see Definition 2.1).

3.10. Let f be a positive functional on a C^* -algebra A , and let $t > 0$. Are the GNS representations (H_f, π_f) and (H_{tf}, π_{tf}) isomorphic? If yes, then construct a unitary isomorphism explicitly.

3.11. Let A be a C^* -algebra, let f be a positive functional on A , and let $\pi_f: A \rightarrow \mathcal{B}(H_f)$ be the GNS representation associated to f . Denote by Λ_f the canonical map from A to H_f , $\Lambda_f(a) = a + N_f$, and let $x_f \in H_f$ be the canonical cyclic vector for π_f uniquely determined by $\langle \Lambda_f(a) | x_f \rangle = f(a)$ ($a \in A$). Describe the quadruple $(H_f, \pi_f, \Lambda_f, x_f)$ explicitly in the following cases:

(a) $A = C_0(X)$, where X is a locally compact Hausdorff topological space, and $f(a) = \int_X a d\mu$, where μ is a finite positive Radon measure on X .

(b) $A = M_n$, $f(T) = \frac{1}{n} \text{Tr}(T)$.

(c) $A = \mathcal{K}(H)$, where H is a Hilbert space, and $f(a) = \langle ah | h \rangle$, where $h \in H \setminus \{0\}$.

(d) $A = C_r^*(G)$, where G is a discrete group, and f is uniquely determined by $f(a) = a(e)$ for $a \in \ell^1(G)$ (see Exercise 3.9).

3.12. Let A be a C^* -algebra, and let $\pi_f: A \rightarrow \mathcal{B}(H_f)$ be the GNS representation of A associated to a positive functional f on A . Denote by $x_f \in H_f$ the canonical cyclic vector for π_f (see Exercise 3.11). Prove that x_f is uniquely determined by $\Lambda_f(a) = \pi_f(a)x_f$ ($a \in A$).

3.13. Let A and f be as in Exercise 3.11 (a). Describe all cyclic vectors $x \in H_f$ for π_f . Give a necessary and sufficient condition for a cyclic vector $x \in H_f$ to satisfy $\langle \pi_f(a)x | x \rangle = f(a)$ ($a \in A$).

3.14. Let A be a C^* -algebra, let f be a positive functional on A , and let π_f denote the GNS representation of A associated to f .

(a) Show that $\text{Ker } \pi_f$ is the largest two-sided ideal of A contained in $\text{Ker } f$.

(b) We say that f is *faithful* if $f(a) = 0$, $a \geq 0$ implies that $a = 0$. Show that if f is faithful, then π_f is faithful (i.e., $\text{Ker } \pi_f = 0$).

(c) Is the converse of (b) true?

3.15. Let A be a separable C^* -algebra. Prove that there exists a state f on A such that the associated GNS representation π_f is faithful.