Def (A,Δ) = CQG; E = fin-dim vec. space.
A corepresentation of (A,Δ) on E is an invertible $U \in L(E)\otimes A$ s.t.

$$(1\otimes\Delta)(U) = U_{12}U_{13} \quad \text{in } L(E)\otimes A\otimes_* A$$

$$U \in L(E)\otimes A \longleftrightarrow \Delta_U \in L(E, E\otimes A) \longleftrightarrow$$
$$\longleftrightarrow \tilde{U} \in \mathrm{End}_A(E\otimes A)$$

Recall: A^* is a Banach alg under convolution:
$f, g \in A^* \qquad f*g = (f\otimes g)\Delta.$

U = fd corep of (A,Δ) on E.
Define $\pi_U : A^* \to L(E)$, $\pi_U(f) = (1_{L(E)}\otimes f)(U)$

Prop. π_U is a rep. of A^* on E.

Lemma/exer. $\forall\, U, V \in L(E)\otimes A$, $\forall\, f, g \in A^*$
$$(1\otimes f\otimes g)(U_{12}V_{13}) = (1\otimes f)(U)\cdot(1\otimes g)(V)$$

Proof of Prop. $f, g \in A^*$

$$\pi_U(f*g) = (1\otimes(f*g))(U) = (1\otimes(f\otimes g)\Delta)(U) =$$
$$= (1\otimes f\otimes g)(1\otimes\Delta)(U) = (1\otimes f\otimes g)(U_{12}U_{13}) \overset{L}{=}$$
$$\overset{L}{=} (1\otimes f)(U)\cdot(1\otimes g)(U) = \pi_U(f)\,\pi_U(g) \qquad \square$$

Prop. U, V = fd coreps of (A, Δ). Then

$$\mathrm{Hom}(U,V) = \mathrm{Hom}(\pi_U, \pi_V)$$

Lemma/exer 1 $\forall f \in A^* \ \forall U \in L(E) \otimes A$

$$(1_{L(E)} \otimes f)(U) = (1_E \otimes f) \circ \Delta_U$$

$$E \xrightarrow[\Delta_U]{} E \otimes A \xrightarrow[1_E \otimes f]{} E.$$

Lemma/exer 2.

X = vec. space, Y = normed space. Then $\forall\, 0 \neq z \in X \otimes Y \ \exists f \in Y^*$ s.t. $(1_X \otimes f)(z) \neq 0$.

Proof of Prop. Let $T \in L(E, F)$

(U = corep on E, V = corep on F)

$$T \in \mathrm{Hom}(U,V) \iff (T \otimes 1)\Delta_U = \Delta_V T \overset{L2}{\iff}$$

$$\underset{L2}{\iff} \forall f \in A^* \quad (1 \otimes f)(T \otimes 1)\Delta_U = (1 \otimes f)\Delta_V T$$

$$(1 \otimes f)(T \otimes 1)\Delta_U = T(1_E \otimes f)\Delta_U \overset{L1}{=} T\pi_U(f)$$

$$(1 \otimes f)\Delta_V T \overset{L1}{=} \pi_V(f) T$$

$$\iff T \in \mathrm{Hom}(\pi_U, \pi_V)$$

$\square$

Prop. U = fd corep of (A, Δ) on E;

$F \subset E$ subspace; $P: E \to E$ a proj. onto F.

TFAE:

(1) $(P \otimes 1)U(P \otimes 1) = U(P \otimes 1)$ in $L(E) \otimes A$.

(2) $\tilde{U}(F\otimes A)\subset F\otimes A$

(3) $\Delta_U(F)\subset F\otimes A$.

Proof: Observe: $P\otimes 1_A$ is a proj. of $E\otimes A$ onto $F\otimes A$.

(1) $\iff (P\otimes 1)\tilde{U}(P\otimes 1)=\tilde{U}(P\otimes 1)$ in $\mathrm{End}_A(E\otimes A)$

$\iff (P\otimes 1)\tilde{U}=\tilde{U}$ on $F\otimes A \iff$ (2)

(2) $\iff$ (3) follows from $\Delta_U(x)=\tilde{U}(x\otimes 1)$,

$\tilde{U}(x\otimes a)=\Delta_U(x)a$. $\square$

Def F is a *U-invariant subspace* $\iff$

$\iff$ U satisfies any (hence all) of the conditions of Prop.

Example $A=C(G)$

$U\in L(E)\otimes A$ is a rep. $U: G\to GL(E)$.

$\Delta_U: E\longrightarrow E\otimes A\cong C(G,E)$;

$(\Delta_U v)(x)=U(x)v \quad (x\in G)$.

F is U-inv $\iff \Delta_U(F)\subset F\otimes A\cong C(G,F)$

$\iff U(x)F\subset F \quad \forall x\in G$

$\iff U(x)F=F \quad \forall x\in G$

$\longmapsto$ subrepresentation: $x\longmapsto U(x)|_F$.

Is there a quantum analog? YES (not obvious!)

Prop. $F \subset E$ is U-inv $\iff$ F is π_U-inv.

Proof. Observe: $\forall f \in A^*$

$\mathbb{1}_{L(E)} \otimes f : L(E) \otimes A \to L(E)$ is an $L(E)$-bimod. morphism.

Let $P: E \to E$ be a proj. onto F.

F is π_U-inv $\iff$ $\forall f \in A^*$

$$P \pi_U(f) P = \pi_U(f) P = (\mathbb{1} \otimes f)(U) P = (\mathbb{1} \otimes f)(U(P \otimes 1))$$

$$P \pi_U(f) P = (\mathbb{1} \otimes f)((P \otimes 1) U (P \otimes 1))$$

$\underset{L}{\iff} (P \otimes 1) U (P \otimes 1) = U(P \otimes 1) \iff F$ is U-inv. $\square$

Def. U = fd corep of (A, Δ) on E.

U is <u>irreducible</u> $\iff$ 0 and E are the only U-inv. subspaces of E

Cor. U is irr. $\iff$ π_U is irr.

Prop (<u>Schur's lemma</u>)

U, V = fd irr. coreps of (A, Δ). Then

(1) If $U \not\cong V$, then $\mathrm{Hom}(U, V) = 0$.

(2) If $U \cong V$, then $\mathrm{Hom}(U, V)$ is 1-dim.

(3) $\mathrm{End}(U) \cong \mathbb{C}\mathbb{1}_E$

Proof Apply the "classical" Schur's lemma to π_U. $\square$

Thm A fd corep. of a CQG is unitarizable.
More exactly: if V is a fd corep of (A,Δ) on a fd Hilb. space H, then $\exists$ a unitary corep U of (A,Δ) on H s.t. $U \cong V$.

Lemma/exer 1. A, B = unital alg, $h: A \to \mathbb{C}$ lin., $X, Y, Z \in B \otimes A \otimes A$. Then
$$(1 \otimes h \otimes 1)(X_{13} Y_{12} Z_{13}) = X\big(((1 \otimes h)(Y) \otimes 1\big) Z.$$

Lemma/exer 2 (A,Δ) = CQG, $V \in L(E) \otimes A$ a fd corep of A; $T \in L(E,F)$ invertible.
Let $U = (T \otimes 1) V (T^{-1} \otimes 1) \in L(F) \otimes A$
Then U is a corep of A on F, and T is an isom $V \xrightarrow{\sim} U$.

Proof of Thm Let $T = (1 \otimes h)(V^* V) \in \mathcal{B}(H)$
$V^* V \geq \varepsilon \cdot 1_{\mathcal{B}(H) \otimes A}$ for some $\varepsilon > 0$. $\Longrightarrow$
$\Longrightarrow T \geq \varepsilon 1_{\mathcal{B}(H)} \Longrightarrow T$ is invertible.
Let $U = (T^{1/2} \otimes 1) V (T^{-1/2} \otimes 1) \in \mathcal{B}(H) \otimes A$.
L2 $\Rightarrow U$ is a corep. of (A,Δ) on H, $U \cong V$.
$$U^* U = (T^{-1/2} \otimes 1) V^* (T^{1/2} \otimes 1)(T^{1/2} \otimes 1) V (T^{-1/2} \otimes 1)$$
$$U^* U = 1 \iff V^* (T \otimes 1) V = T \otimes 1 \qquad (*)$$

V is a corep $\Rightarrow$

$$\Rightarrow (1\otimes\Delta)(V^*V) = V_{13}^* V_{12}^* V_{12} V_{13} \text{ in } \mathcal{B}(H)\otimes A\otimes_* A$$

Apply $1\otimes h\otimes 1$:

$$(1\otimes h\otimes 1)(1\otimes\Delta)(V^*V) = (1\otimes \eta h)(V^*V) =$$
$$= (1\otimes\eta)(1\otimes h)(V^*V) = T\otimes 1.$$

$$(1\otimes h\otimes 1)(V_{13}^* \underbrace{V_{12}^* V_{12}}_{(V^*V)_{12}} V_{13}) \underset{L1}{=} V^*((1\otimes h)(V^*V)\otimes 1)V$$
$$= V^*(T\otimes 1)V$$

$$\Rightarrow T\otimes 1 = V^*(T\otimes 1)V \underset{(*)}{\Longrightarrow} U \text{ is unitary. } \square$$

Remark

Standard fact: G = compact group,
$\pi: G \to GL(H)$ rep. of G on a f.d. Hilb. space H
$\Rightarrow \pi \simeq$ a unitary rep

Standard proof: define a new inner prod

$$\langle u|v\rangle_G = \int_G \langle \pi(x)u|\pi(x)v\rangle \, d\mu(x)$$

$\Rightarrow \pi$ is unitary wrt $\langle\cdot|\cdot\rangle_G$.

Exer. Show that, if $A = C(G)$, then our proof and the stand. proof are "essentially the same".

Cor. 1 $(A,\Delta) = \mathrm{CQG}$; $u \in M_n(A)$ is a matrix corep of A $\quad (\Delta u_{ij} = \sum_k u_{ik} \otimes u_{kj})$

Then $\exists\, T \in GL(n,\mathbb{C})$ s.t. TuT^{-1} is unitary.

Cor. 2 $(A,\Delta,u) = \mathrm{CMQG}$. Then $\exists\, T \in GL(n,\mathbb{C})$ s.t. $u' = TuT^{-1}$ is unitary. Moreover, (A,Δ,u') is a CMQG.

Proof exer. $\square$.

Convention: identify (A,Δ,u) with (A,Δ,u').

Prop. $(A,\Delta,u) =$ comm. CMQG; $u \in M_n(A)$ is unitary. Let $G = \operatorname{Max} A$. Then:

(1) The map $G \to U(n)$, $\chi \mapsto (\chi(u_{ij}))$, is a top. isom. of G onto a closed subgroup of $U(n)$.

(2) The Gelfand isom. $\Gamma : A \xrightarrow{\sim} C(G)$ satisfies

$$\Gamma(u_{ij})(x) = x_{ij} \quad \forall x \in G \subset U(n)$$

Proof: exer (see the proof of the univ. prop. of $C(SU(2))$)

Facts: (1) $G \subset U(n)$ closed subgroup $\Longrightarrow$
$\Rightarrow G$ is a comp. Lie group

(2) $G =$ comp. Lie group $\Rightarrow G$ is top. isomorphic to a closed subgroup of $U(n)$ (for some n).

(3) G, H = Lie groups, $f: G \to H$ cont. group hom
$\Rightarrow f$ is smooth.

Hence $\exists$ an equivalence

$$\{\text{Comp. Lie groups}\}^{op} \rightleftarrows \{\text{Comm. CMQGs}\}.$$

<u>Philosophy</u>: CMQG = "compact quantum Lie group".