$(A, \Delta) = CQG \qquad h \in S(A)$.

<u>Def</u> h is a <u>Haar state</u> $\iff \forall \omega \in A^* \ (\omega \geq 0)$

$$\omega * h = h * \omega = \omega(1) h$$

<u>Thm</u> h exists and is unique.

<u>Lemma 2</u> $h \in S(A)$, $\omega, \nu \in A^*$, $0 \leq \nu \leq \omega$.

If $\omega * h = h * \omega = \omega(1) h$, then $\nu * h = h * \nu = \nu(1) h$.

<u>Proof</u> $(\nu * h = \nu(1) h)$. We may assume $\omega(1) = 1$.

$\forall b \in A$ let $c(b) = c = (1 \otimes h) \Delta b \in A$.

<u>Sublemma 1</u> $\forall f \in A^* \quad (1 \otimes f * h) \Delta b = (1 \otimes f) \Delta c$.

<u>Proof</u>: exer (coass. of Δ) $\square$

<u>Sublemma 2</u> $\nu * h = \nu(1) h \iff$

$$\forall a, b \in A \qquad (a \otimes 1)(\Delta c - c \otimes 1) \in \operatorname{Ker}(h \otimes \nu) \qquad (1)$$

<u>Proof</u>. $\nu * h = \nu(1) h \iff$

$\iff h \otimes \nu * h = \nu(1) h \otimes h$ on $\underbrace{(A \otimes 1)\Delta(A) \subset A \otimes_* A}_{\text{(dense)}}$

$$(\nu(1) h \otimes h)((a \otimes 1)\Delta b) = \nu(1) h (1 \otimes h)((a \otimes 1)\Delta b) =$$

$$= \nu(1) h(a \cdot (1 \otimes h)(\Delta b)) = \nu(1) h(ac) = (h \otimes \nu)(ac \otimes 1) \quad (2)$$

$$(h \otimes \nu * h)((a \otimes 1)\Delta b) = h(1 \otimes \nu * h)((a \otimes 1)\Delta b)$$

$$= h(a \cdot (1 \otimes \nu * h)(\Delta b)) \underset{(S1)}{=} h(a \cdot (1 \otimes \nu)(\Delta c))$$

$$= h(1\otimes\nu)((a\otimes 1)\Delta c) = (h\otimes\nu)((a\otimes 1)\Delta c) \quad (3).$$

Hence $\nu * h = \nu(1)h \iff (2)=(3) \iff$

$\iff (a\otimes 1)(\Delta c - c\otimes 1) \in Ker(h\otimes\nu)$ $\square$

Let $L_{h\otimes\nu} = \{u\in A\otimes_* A : (h\otimes\nu)(u^*u)=0\}$

$= \{u\in A\otimes_* A : (h\otimes\nu)(vu)=0 \ \forall v\in A\otimes_* A\}$

<u>Sublemma 3</u> If $\forall b\in A$ $\Delta c - c\otimes 1 \in L_{h\otimes\omega}$, then (1) holds.

<u>Proof</u> $0\le\nu\le\omega \Rightarrow L_{h\otimes\omega} \subset L_{h\otimes\nu} \subset Ker(h\otimes\nu)$; $L_{h\otimes\omega}$ is a left ideal of $A\otimes_* A$ $\square$

<u>End of Proof of L2</u>

$$(h\otimes\omega)((\Delta c - c\otimes 1)^*(\Delta c - c\otimes 1)) =$$

$$= \underbrace{(h\otimes\omega)(\Delta(c^*c))}_{\alpha_1} - \underbrace{(h\otimes\omega)((c^*\otimes 1)\Delta c)}_{\alpha_2}$$

$$- \underbrace{(h\otimes\omega)(\Delta(c^*)(c\otimes 1))}_{\alpha_3} + \underbrace{(h\otimes\omega)(c^*c\otimes 1)}_{\alpha_4} \quad (4)$$

$\alpha_4 = h(c^*c)$

$\alpha_1 = (h*\omega)(c^*c) = h(c^*c) = \alpha_4$.

$\alpha_2 = h(1\otimes\omega)((c^*\otimes 1)\Delta c) = h(c^*(1\otimes\omega)\Delta c) \overset{(S1)}{=}$

$= h(c^*(1\otimes\omega * h)\Delta b) = h(c^*\underbrace{(1\otimes h)\Delta b}_{c}) = \alpha_4$.

$\alpha_3 = \overline{\alpha_2} = \overline{\alpha_4} = \alpha_4 \Rightarrow \alpha_1 = \ldots = \alpha_4 \Rightarrow (4) = 0$

$\Rightarrow \Delta c - c\otimes 1 \in L_{h\otimes\omega}$ $\square$.

Corepresentations of CQGs

Notation E, F = fin-dim vec. spaces.

$L(E,F) = \mathrm{Hom}_{\mathbb{C}}(E,F)$; $L(E) = L(E,E)$.

The standard top on E = the top. gener. by a norm.

$GL(E) = \{\text{invertible } T \in L(E)\} \subset L(E)$ is a top. group.

Def G = top. group. A representation of G on E is a cont. hom. $\pi : G \to GL(E)$.

Def H = f.d. Hilb. space, π = rep. of G on H.
π is unitary $\iff$ $\pi(x)$ is unitary $\forall x \in G$.

Def (π_1, E_1) and (π_2, E_2) f.d. reps of G.
A morphism (intertwining map) $\pi_1 \to \pi_2$ is a lin. $T : E_1 \to E_2$ s.t. $T\pi_1(x) = \pi_2(x) T$ $\forall x \in G$.

$$\mathrm{Hom}(\pi_1, \pi_2) = \{\text{morphisms } \pi_1 \to \pi_2\} = \mathrm{Hom}_G(E_1, E_2).$$

Rep. Th. of comp. groups (and of CQGs!)

+ (1) Basic theory
- (2) Peter-Weyl theory
+ (3) Tannaka-Krein theory

Notation A, B = unital alg

$I_{12}, I_{13} : B \otimes A \to B \otimes A \otimes A$;

$I_{12} : b \otimes a \mapsto b \otimes a \otimes 1$;

$I_{13} : b \otimes a \mapsto b \otimes 1 \otimes a$.

$\forall\, U \in B \otimes A \quad U_{12} = I_{12}(U)$; $U_{13} = I_{13}(U)$

(leg-numbering notation)

(A, Δ) = CQG; E = f.d. vec. space

Def A corepresentation of (A, Δ) on E is an invertible $U \in L(E) \otimes A$ s.t.

$$(1 \otimes \Delta)(U) = U_{12} U_{13} \quad \text{in } L(E) \otimes (A \otimes_* A)$$

H = f.d. Hilb. space; note: $L(H) = \mathcal{B}(H)$ is a C^*-alg.

Def A corep U of A on H is unitary $\Longleftrightarrow$

$\Longleftrightarrow$ U is a unitary elem of $\mathcal{B}(H) \otimes A$.

Example/exer. $A = C(G)$; G = comp. group

$L(E) \otimes A \cong C(G, L(E))$.

An invertible $U \in L(E) \otimes A$ is a cont. map

$U : G \to GL(E)$.

$L(E) \otimes (A \otimes_* A) \cong C(G \times G, L(E))$

$[(1 \otimes \Delta)(U)](x,y) = U(xy)$; $U_{12}(x,y) = U(x)$;

$U_{13}(x,y) = U(y)$. Hence

$(1\otimes\Delta)U = U_{12}U_{13} \iff U(xy) = U(x)U(y) \ \forall x,y$

$$\begin{array}{ccc} \{\text{f.d reps of } G\} & \rightleftarrows & \{\text{f.d. coreps of } C(G)\} \\ \cup & & \cup \\ \{\text{unitary reps}\} & \rightleftarrows & \{\text{unitary coreps}\} \end{array} \qquad (D)$$

E, F = f.d. vec. spaces, A = unital alg.

<u>Observe:</u> $L(E,F)\otimes A \cong L(E, F\otimes A) \cong$

$\cong \mathrm{Hom}_A(E\otimes A, F\otimes A)$

$T\otimes a \mapsto (v \mapsto T(v)\otimes a)$;

$S \mapsto (v\otimes a \mapsto S(v)\cdot a)$

$U \in L(E)\otimes A \longleftrightarrow (\Delta_U : E \to E\otimes A) \longleftrightarrow$

$\longleftrightarrow (\tilde{U} : E\otimes A \to E\otimes A)$

<u>Example</u> (<u>exer</u>). $A = C(G)$; $L(E)\otimes A = C(G, L(E))$

$E\otimes A = C(G,E)$

$\Delta_U : E \to C(G,E)$, $(\Delta_U v)(x) = U(x)v$.

$\tilde{U} : C(G,E) \to C(G,E)$; $(\tilde{U}f)(x) = U(x)f(x)$.

Exer. $(A,\Delta) = CQG$; E = f.d. vec. space, $U \in L(E)\otimes A$. Then $(1\otimes\Delta)(U) = U_{12}U_{13}$

$\Longleftrightarrow$
$$\begin{tikzcd} E \arrow[r, "\Delta_U"] \arrow[d, "\Delta_U"'] & E\otimes A \arrow[d, "\Delta_U\otimes 1_A"] \\ E\otimes A \arrow[r, "1_E\otimes\Delta"'] & E\otimes(A\otimes_* A) \end{tikzcd}$$
commutes

Def $(A,\Delta) = CQG$; U, V = f.d. coreps of (A,Δ) on E, F, resp.

A <u>morphism</u> from U to V (<u>intertwining map</u>) is a lin $T: E\to F$ s.t.

$$(T\otimes 1)\tilde{U} = \tilde{V}(T\otimes 1) \text{ in } \mathrm{Hom}_A(E\otimes A, F\otimes A)$$

Equivalently:
$$\begin{tikzcd} E \arrow[r, "\Delta_U"] \arrow[d, "T"'] & E\otimes A \arrow[d, "T\otimes 1"] \\ F \arrow[r, "\Delta_V"'] & F\otimes A \end{tikzcd}$$
commutes

Example: $A = C(G)$

$[(T\otimes 1)\Delta_U w](x) = T(U(x)w)$ (exer)

$[\Delta_V Tw](x) = V(x)Tw.$

T is a morph of coreps $\Longleftrightarrow$ T is a morph. of reps.
Hence (D) is an isomorphism of categories.

$(A,\Delta) = $ CQG; $E = $ f.d. vec. space;

$(e_1,\dots,e_n) = $ basis of E;

$L(E) \cong M_n$; $L(E)\otimes A \cong M_n \otimes A \cong M_n(A)$

$U \in L(E)\otimes A \longleftrightarrow u = (u_{ij}) \in M_n(A)$.

__Exer.__ U is a corep $\Longleftrightarrow$ u is invertible,

and $\Delta u_{ij} = \sum_k u_{ik} \otimes u_{kj} \quad \forall i,j. \quad (*)$

__Def.__ A __matrix corep.__ (resp, a __unitary matrix corep__) is an invertible (resp. unitary) $u \in M_n(A)$ satisfying $(*)$.

__Example.__ $(A,\Delta,u) = $ CMQG. $\Rightarrow$ u is a matrix corep. (the __fundamental__ corep. of A).