Slice maps on minimal C^*-tensor products

Notation H = Hilb. space, $x, y \in H$

$x \odot y \in \mathcal{B}_0(H)$, $(x \odot y)(z) = \langle z | y \rangle x$.

Prop $\forall\, T \in \mathcal{B}(H)$

(1) $T \cdot (x \odot y) = Tx \odot y$.

(2) $(x \odot y) \cdot T = x \odot T^* y$.

(3) $(x \odot y) T (z \odot u) = \langle Tz | y \rangle\, x \odot u$.

(4) $\| x \odot y \| = \|x\| \|y\|$.

Proof: exer.

Thm $A, B = C^*$-alg, $f \in A^*$. Then $\exists$ a bdd lin. oper. $f \otimes_* 1 : A \otimes_* B \to B$ uniquely def'd by $a \otimes b \mapsto f(a) b$. $(a \in A,\ b \in B)$. Moreover, $\| f \otimes_* 1 \| = \| f \|$.

Proof (Helemskii 2009).

Consider $f \otimes 1 : A \otimes B \to B$, $a \otimes b \mapsto f(a) b$.

It suff. to show that $f \otimes 1$ is bdd wrt $\| \cdot \|_*$, and that $\| f \otimes 1 \| \leq \| f \|$.

$B \hookrightarrow \mathcal{B}(H)$.

$$\begin{array}{ccc} A\otimes B & \xrightarrow{f\otimes 1} & B \\ \big\downarrow \text{isometr.} & & \big\downarrow \text{isometr.} \\ A\otimes \mathcal{B}(H) & \xrightarrow{f\otimes 1} & \mathcal{B}(H) \end{array}$$

$\Rightarrow$ we may assume that $B=\mathcal{B}(H)$.

Let $F=f\otimes 1$.

Observe: B and $A\otimes B$ are B-bimodules, and F is a morphism of B-bimodules

$b\cdot(a\otimes b')=a\otimes bb'$

Let $u\in A\otimes B$.

$\|F(u)\| = \sup\{|\langle F(u)x|y\rangle| : \|x\|\le 1, \|y\|\le 1\}$.

Take $x,y,h\in H$, $\|x\|\le 1$, $\|y\|\le 1$, $\|h\|=1$.

Let $p=h\odot h$.

$\langle F(u)x|y\rangle p = (h\odot y)\, F(u)\,(x\odot h) =$

$= F((h\odot y)\cdot u\cdot (x\odot h))$.

<u>Exer.</u> $(h\odot y)\cdot u\cdot(x\odot h)\in A\otimes \mathbb{C}p$.

Hence $(h\odot y)\cdot u\cdot (x\odot h) = b\otimes p$ $(b\in A)$

$\Rightarrow |\langle F(u)x|y\rangle| = \|\langle F(u)x|y\rangle p\| =$

$=\|F(b\otimes p)\| = \|f(b)p\| = |f(b)| \le \|f\|\,\|b\| =$

$=\|f\|\,\|b\otimes p\| \le \|f\|\,\|u\|$ $\Rightarrow$ F is bdd, and

$\|F\|\le\|f\|$. $\square$.

Notation. We write $f \otimes 1$ for $f \otimes_* 1$.

Cor. $f \in A^*$, $g \in B^* \Rightarrow \exists\, f \otimes g \in (A \otimes_* B)^*$ uniquely det'd by $a \otimes b \mapsto f(a)g(b)$ $(a \in A,\ b \in B)$

Proof $f \otimes g = f \circ (1 \otimes g) = g \circ (f \otimes 1)$ $\square$

Exer. $H = \ell^2$; $\theta : \mathcal{K}(H) \to \mathcal{K}(H)$,
$\theta(T) = T^t$, where $\langle T^t e_i | e_j \rangle = \langle T e_i | e_j \rangle$.
Prove: θ is bdd, but
$\theta \otimes 1 : \mathcal{K}(H) \otimes \mathcal{K}(H) \to \mathcal{K}(H) \otimes \mathcal{K}(H)$ is unbdd w.r.t. $\|\cdot\|_*$.

The Haar state

G = loc. comp. group, μ = pos. Radon meas on G (not necessarily finite).

Def. μ is <u>left invariant</u> (resp. <u>right inv.</u>) $\Leftrightarrow$
$\Leftrightarrow \forall$ Borel $B \subset G$ $\forall x \in G$ $\mu(xB) = \mu(B)$
(resp. $\mu(Bx) = \mu(B)$)
If $\mu \neq 0$, then μ is a left (resp right) <u>Haar measure</u> on G.

Examples: the counting measure on a discrete G; the Lebesgue meas on $\mathbb{R}^n$ or on $\mathbb{T}^n$.

Thm. (A. Haar, J. von Neumann, A. Weil)

(1) $\exists$ left and right Haar measures on G.

(2) μ, ν = left Haar measures on G $\Rightarrow \exists\, c>0$ s.t. $\nu = c\mu$. Similarly for right Haar meas.

(3) If G is compact, μ = a pos. Radon meas on G. μ is left inv $\Longleftrightarrow$ μ is right inv.

Def G = compact gr, μ = Haar meas on G. μ is <u>normalized</u> $\Longleftrightarrow$ $\mu(G)=1$.

Such μ is unique.

G = comp. group, μ = pos. Radon meas on G;

$h \in C(G)^*$, $h(f) = \int_G f\,d\mu$.

$\forall\, x \in G$ define a Rad. meas. $L_x\mu$ on G by

$(L_x\mu)(B) = \mu(x^{-1}B)$

μ is left inv $\Longleftrightarrow$ $\forall\, x\in G$ $\mu = L_x\mu$ $\Longleftrightarrow$

$\Longleftrightarrow$ $\forall\, f \in C(G)$ $\int f\,d\mu = \int f\,d(L_x\mu)$

$$h(f) = \int f\,d\mu, \qquad \int f\,d(L_x\mu) \overset{\text{(exer)}}{=} \int f(xy)\,d\mu(y) = (1\otimes h)\Delta f.$$

Notation A = unital alg. $\eta: \mathbb{C} \to A$, $\eta(1) = 1_A$.

Hence: μ is left inv $\Leftrightarrow (1\otimes h)\Delta = \eta h$.
μ is right inv $\Leftrightarrow (h\otimes 1)\Delta = \eta h$.

<u>Def</u> $(A,\Delta) = CQG$; $h\in A^*$, $h\geq 0$
h is <u>left inv</u> $\Leftrightarrow (1\otimes h)\Delta = \eta h$
h is <u>right inv</u> $\Leftrightarrow (h\otimes 1)\Delta = \eta h$.

<u>Def</u> The <u>convolution</u> of $f,g\in A^*$ is $f*g\in A^*$,
$f*g = (f\otimes g)\Delta$.

<u>Special case</u>: convolution of measures
G = comp. group; $\mu,\nu\in M(G)$

$$\int_G f\, d(\mu*\nu) = \int_{G\times G} f(xy)\,d\mu(x)\,d\nu(y) \quad (f\in C(G)).$$

<u>Exer</u> (1) $(A^*, *)$ is a Ban. algebra.
(2) $f,g\geq 0 \Rightarrow f\otimes g\geq 0 \Rightarrow f*g\geq 0$.
(3) $f,g\in S(A) \Rightarrow f\otimes g\in S(A\otimes_* A) \Rightarrow f*g\in S(A)$.

<u>Prop</u>. $h\in A^*$, $h\geq 0$. TFAE:
(1) h is left (resp. right) inv;
(2) $\forall\, \omega\in A^*$ $\omega*h = \omega(1)h$ (resp. $h*\omega = \omega(1)h$)
(3) $\forall\, \omega\in A^*$, $\omega\geq 0$ we have (2).

<u>Proof</u> $\omega*h = (\omega\otimes h)\Delta = \omega(1\otimes h)\Delta$.
$\omega(1)h = \omega\eta h$.

Hence (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (3) clear.

Lemma (exer). $\forall$ C^*-alg A $\forall a \in A \setminus \{0\}$

$\exists\, \omega \in S(A)$ s.t. $\omega(a) \neq 0$.

Hint: $a = b + ic$, $b, c \in A_{sa}$.

(3) $\Rightarrow$ (1) $\omega((1 \otimes h)\Delta - \eta h) = 0$ $\forall \omega \in S(A)$

$\underset{L.}{\Longrightarrow}$ $(1 \otimes h)\Delta - \eta h = 0 \Rightarrow h$ is left inv. $\square$

Thm (Woronowicz).

A = CQG.

(1) $\exists\, h \in S(A)$ s.t. h is left inv and right inv.

(2) If $f \in S(A)$ s.t. f is left or right inv, then $f = h$.

Def h is the Haar state on A.

Lemma 1. $\forall \omega \in S(A)$ $\exists\, h \in S(A)$ s.t.

$$\omega * h = h * \omega = h.$$

Proof. Let $h_n = \frac{1}{n}(\omega + \omega * \omega + \dots + \omega^{*n}) \in S(A)$.

$(S(A), wk^*)$ is compact $\Rightarrow \exists$ a subnet $h_{n(\lambda)} \to h \in S(A)$

$$\omega * h_n - h_n = h_n * \omega - h_n = \frac{1}{n}(\underbrace{\omega^{*(n+1)} - \omega}_{bdd}) \Longrightarrow$$

$\Rightarrow \omega * h - h = h * \omega - h = 0.$ $\square$

Lemma 2. $h \in S(A)$; $\omega, \nu \in A^*$, $0 \le \nu \le \omega$.
If $\omega * h = h * \omega = \omega(1) h$, then $\nu * h = h * \nu = \nu(1) h$.

Proof of Thm (modulo L2)
$\forall\, \omega \in A^*$, $\omega \geq 0$, let $K_\omega = \{h \in S(A) : \omega * h = h * \omega = \omega(1) h\}$

Exer: convolution on A^* is separately wk*-cont.
Hence $K_\omega \subset S(A)$ is wk*-closed $\Rightarrow$ K_ω is comp.
$\bigcap_{i=1}^{n} K_{\omega_i} \supset K_{\omega_1 + \ldots + \omega_n}$ by L2.
By L1, $K_\omega \neq \emptyset$ $\forall\, \omega \geq 0$.
Hence $\bigcap_{\omega \geq 0} K_\omega \neq 0$. Take any $h \in \bigcap_{\omega \geq 0} K_\omega$
$\Rightarrow$ h is left and right inv.
If $f \in S(A)$ is left inv, then $f = f * h = h$. □