Def A <u>compact quantum group</u> is a unital C^*-bialg (A,Δ) s.t. $\Delta(A)(1\otimes A)$ and $\Delta(A)(A\otimes 1)$ are dense in $A\otimes_* A$

$$\{\text{Comp. groups}\}^{op} \rightleftarrows \{\text{Comm. CQGs}\}$$

Compact matrix quantum groups. The quantum $SU(2)$

Prop. $C(SU(2)) \cong C^*\left(\alpha, \gamma \,\middle|\, \begin{pmatrix} \alpha & -\gamma^* \\ \gamma & \alpha^* \end{pmatrix} \text{ is unitary}\right)$.

Proof $SU(2) = \{g \in M_2(\mathbb{C}) : \bar{g}^T = g^{-1},\ \det g = 1\}$

$= \{g_{z,w} : z, w \in \mathbb{C},\ |z|^2 + |w|^2 = 1\}$;

where $g_{z,w} = \begin{pmatrix} z & -\bar{w} \\ w & \bar{z} \end{pmatrix}$.

Define $\alpha, \gamma \in C(SU(2))$, $\alpha(g_{z,w}) = z$, $\gamma(g_{z,w}) = w$.

$u = \begin{pmatrix} \alpha & -\gamma^* \\ \gamma & \alpha^* \end{pmatrix} \in M_2(C(SU(2))) \cong C(SU(2), M_2)$

$u \longleftrightarrow$ incl. map $SU(2) \hookrightarrow M_2$.

$\Rightarrow u$ is unitary.

Suppose A is a unital C^*-alg, $a, b \in A$ s.t.

$v = \begin{pmatrix} a & -b^* \\ b & a^* \end{pmatrix}$ is unitary.

Exer. v is unitary $\iff$ a, b, a^*, b^* commute, and
$(v^*v = vv^* = 1)$ $\qquad a^*a + b^*b = 1$.

$B = C^*_A(a,b)$. Exer $\Rightarrow$ B is comm.

$X = \mathrm{Max}\, B$. $\forall \chi \in X = \hat{B}$ $|\chi(a)|^2 + |\chi(b)|^2 = 1$.

consider $f: X \to SU(2)$, $f(\chi) = g_{\chi(a),\chi(b)}$.

$$\begin{tikzcd} C(SU(2)) \arrow[r, "f^\circ"] \arrow[rrr, bend right, "\varphi"'] & C(X) \arrow[r, "\Gamma_B^{-1}", "\sim"'] & B \arrow[r, hook] & A \end{tikzcd}$$

Exer. $\varphi(\alpha) = a$, $\varphi(\gamma) = b$.

The uniqueness of φ:

S-Weier $\Rightarrow$ the $*$-subalg of $C(SU(2))$ gener. by α, γ is dense in $C(SU(2))$. $\square$.

Def $q \in [-1,1]$, $q \neq 0$.

The alg. of "continuous functions" on the quantum SU(2) group (briefly: the quantum SU(2)) is $C_q(SU(2)) = SU_q(2) = C^*\left(\alpha, \gamma \;\middle|\; \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix} \text{ is unitary}\right)$

Observe: if $q = 1$, then $C_q(SU(2)) \cong C(SU(2))$.

Notation $A = *$-alg; $u \in M_n(A)$.

Define $u^T, \bar{u} \in M_n(A)$; $(u^T)_{ij} = u_{ji}$; $\bar{u}_{ij} = u_{ij}^*$.

Observe: $u^* = \bar{u}^T$.

Def A compact matrix quantum group is (A, Δ, u) where A is a unital C^*-alg, $\Delta : A \to A \otimes_* A$ is a unital $*$-hom, $u \in M_n(A)$ s.t.

(1) u and u^T are invertible;

(2) A is gener. by u_{ij} $(1 \le i, j \le n)$ (as a unital C^*-alg)

(3) $\Delta(u_{ij}) = \sum_{k=1}^{n} u_{ik} \otimes u_{kj} \quad \forall i, j.$

(Woronowicz 1991).

Observe: u^T is inv $\Longleftrightarrow \bar{u}$ is inv. (because $(u^T)^* = \bar{u}$)

Example $G \subset U(n)$ closed subgroup;
$A = C(G)$; $u \in M_n(A)$; $u_{ij}(g) = g_{ij}$.
$M_n(A) \cong C(G, M_n)$
$u \longleftrightarrow$ incl. map $G \hookrightarrow M_n$.
$\Rightarrow \forall g \in G$ $u(g)$ and $u(g)^T$ are invertible
$\Rightarrow u, u^T$ are invertible.
St.-Weier. $\Rightarrow$ (2) holds.

$$(\Delta u_{ij})(g, h) = u_{ij}(gh) = \sum_k g_{ik} h_{kj} =$$
$$= \Big(\sum_k u_{ik} \otimes u_{kj}\Big)(g, h) \Rightarrow (C(G), \Delta, u) \text{ is a CMQG.}$$

<u>Thm.</u> (A,Δ,u) is a CMQG $\Rightarrow (A,\Delta)$ is a CQG.

<u>Proof.</u> (2),(3) $\Rightarrow \Delta$ is coass. (exer.)

$B = \{a \in A : a \otimes 1 \in \Delta(A)(1 \otimes A)\}$.

<u>Lemma 1.</u> B is a unital subalg of A.

<u>Proof</u> $a, a' \in B$.
$$a \otimes 1 = \sum_j \Delta(b_j)(1 \otimes c_j)$$
$$a' \otimes 1 = \sum_k \Delta(b'_k)(1 \otimes c'_k).$$

$$aa' \otimes 1 = (a \otimes 1)(a' \otimes 1) = \sum_j \Delta(b_j)(a' \otimes 1)(1 \otimes c_j) =$$
$$= \sum_{j,k} \Delta(b_j)\Delta(b'_k)(1 \otimes c'_k)(1 \otimes c_j) = \sum_{j,k} \Delta(b_j b'_k)(1 \otimes c'_k c_j)$$
$\Rightarrow aa' \in B$. $\square$.

<u>Lemma 2.</u> $u_{ij} \in B \quad \forall i,j$.

<u>Proof</u> Let $v = u^{-1}$.
$$\sum_k \Delta(u_{ik})(1 \otimes v_{kj}) = \sum_{k,\ell} u_{i\ell} \otimes u_{\ell k} v_{kj} = u_{ij} \otimes 1$$
$\Rightarrow u_{ij} \in B$ $\square$.

<u>Lemma 3.</u> $u^*_{ij} \in B \quad \forall i,j$. (similar to L2)

<u>End of the proof of Thm</u>

L2, L3 $\Rightarrow \overline{B} = A$.

$\forall b \in B \ \forall a \in A \quad b \otimes a = (b \otimes 1)(1 \otimes a) \in$

$\in \Delta(A)(1\otimes A)(1\otimes A) = \Delta(A)(1\otimes A) \Rightarrow$

$\Rightarrow B\otimes A \subset \Delta(A)(1\otimes A) \Rightarrow \overline{\Delta(A)(1\otimes A)} = A\otimes_* A$.

Similarly: $\overline{\Delta(A)(A\otimes 1)} = A\otimes_* A$ (exer.) $\square$.

<u>Prop.</u> $q\in[-1,1], q\neq 0$; $u = \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix} \in M_2(C_q(SU(2)))$

(1) The defining relations for $C_q(SU(2))$ are as follows: $\alpha\gamma = q\gamma\alpha$, $\alpha\gamma^* = q\gamma^*\alpha$, $\gamma^*\gamma = \gamma\gamma^*$; $\alpha^*\alpha + \gamma^*\gamma = 1$; $\alpha\alpha^* + q^2\gamma^*\gamma = 1$.

(2) $\exists$ a unique unital $*$-hom

$$\Delta: C_q(SU(2)) \to C_q(SU(2)) \otimes_* C_q(SU(2))$$

s.t. $\Delta(u_{ij}) = \sum_{k=1}^{2} u_{ik}\otimes u_{kj}$

(that is, $\Delta(\alpha) = \alpha\otimes\alpha - q\gamma^*\otimes\gamma$, $\Delta(\gamma) = \gamma\otimes\alpha + \alpha^*\otimes\gamma$)

(3) $(C_q(SU(2)), \Delta, u)$ is a CMQG.

<u>Proof</u> (1), (2) exer.

(3) Let $\lambda\in\mathbb{C}$, $\lambda^2 = q \Rightarrow \begin{pmatrix} 0 & -\lambda \\ \lambda^{-1} & 0 \end{pmatrix} \bar{u} \begin{pmatrix} 0 & \lambda \\ -\lambda^{-1} & 0 \end{pmatrix} = u$.

(exer.) $\square$

The quantum $SL(2)$ and the quantum $SU(2)$

$SL(2) = SL(2,\mathbb{C}) = \{g \in M_2(\mathbb{C}) : \det g = 1\}$.

$\sigma : SL(2) \to SL(2)$, $\sigma(g) = (\bar{g}^T)^{-1}$.

$SU(2) = \{g \in SL(2) : \sigma(g) = g\}$.

$\sigma \in \mathrm{Aut}(SL(2))$; $\sigma^2 = 1$; $d\sigma(e)$ is $\mathbb{C}$-antilin.

$\boxed{SU(2) \text{ is a real form of } SL(2)}$

$\mathcal{O}(SL(2)) \cong \mathcal{O}(M_2)/(\det - 1) =$

$= \mathbb{C}[a,b,c,d]/(ad-bc-1)$

Involution on $\mathcal{O}(SL(2))$: $f^*(g) = \overline{f(\sigma(g))}$.

__Exer.__ $SL(2) \rightleftarrows \{\text{unital characters } \mathcal{O}(SL(2)) \to \mathbb{C}\}$.

$SU(2) \rightleftarrows \{\text{unital } *\text{-characters } \mathcal{O}(SL(2)) \to \mathbb{C}\}$.

Let $r : \mathcal{O}(SL(2)) \to C(SU(2))$, $r(f) = f|_{SU(2)}$.

__Exer.__ $(C(SU(2)), r)$ is a C^*-envelope of $\mathcal{O}(SL(2))$

__Exer.__ The invol. on $\mathcal{O}(SL(2))$ is uniquely det'd by $a^* = d$, $b^* = -c$.

__Def__ $q \in \mathbb{C}\setminus\{0\}$. The alg. of "regular functions" on the __quantum $SL(2)$__ is the unital alg $\mathcal{O}_q(SL(2))$ gener. by a,b,c,d with relations $ab = qba$, $ac = qca$, $cd = qdc$, $bd = qdb$, $bc = cb$, $ad - da = (q - q^{-1})bc$; $ad - qda = 1$.

<u>Observe</u>: if $q=1$, then $\mathcal{O}_q(SL(2)) \cong \mathcal{O}(SL(2))$.

<u>Exer.</u> Suppose $q \in \mathbb{R} \setminus \{0\}$.

(1) $\exists$ an invol. on $\mathcal{O}_q(SL(2))$ uniquely det'd by $a^* = d$, $b^* = -qc$.

(2) Suppose $q \in [-1, 1]$. Then $\exists$ a unique $*$-hom $\mathcal{O}_q(SL(2)) \to C_q(SU(2))$ s.t. $a \mapsto \alpha$, $c \mapsto \gamma$.

(3) $(C_q(SU(2)), r)$ is a C^*-env of $\mathcal{O}_q(SL(2))$.