Recall: $A = *$-alg.

$$\begin{tikzcd} C^*(A) \arrow[r, "\psi"] & B \\ A \arrow[u, "\theta"] \arrow[ur, "\varphi"'] & \end{tikzcd}$$

B C^*-alg $\exists$ a unique ψ.

$$\mathrm{Hom}(A,B) \cong \mathrm{Hom}(C^*(A), B)$$
$$(\forall\ C^*\text{-alg } B)$$

$F = \mathbb{C}\langle x_i, y_i \mid i \in I \rangle \qquad x_i^* = y_i.$

$\{p_j\} \subset F \qquad K \subset F$ gen. by $\{p_j\}$.

$C^*(x_i \mid p_j) = C^*(F/K)$.

Examples of C^*-envelopes

① $A = \mathbb{C}[t^{\pm 1}]$ (Laurent poly) $t^* = t^{-1}$.

Exer. $C^*(A) \cong C^*(u \mid u \text{ is unitary}) \cong C(\mathbb{T})$.

$(\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\})$.

② $A = \mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}] \qquad t_i^* = t_i^{-1}$.

Exer. $C^*(A) \cong C^*_{com}(u_1, \dots, u_n \mid u_i \text{ are unitary})$
$\cong C(\mathbb{T}^n)$.

③ $A = \mathbb{C}[t]$ or $\mathbb{C}[t^{\pm 1}] \qquad t^* = t$.

Exer. $C^*(A)$ does not exist.

(4) $A = \mathbb{C}\langle p, q \mid [p,q] = i1 \rangle$ (<u>Weyl alg.</u>)

$p^* = p$; $q^* = q$.

<u>Exer</u> $C^*(A) = 0$.

(5) (<u>Toeplitz alg</u>)

$C^*(u \mid u^*u = 1) \cong$ the C^*-subalg of $\mathcal{B}(\ell^2)$ gener. by the right shift.

(Coburn's thm). (exer*)

(6) (<u>Quantum 2-torus</u>) (<u>rotation alg.</u>)

$\theta \in \mathbb{R}$

$A_\theta = C^*(u, v \mid u, v \text{ are unitary, } uv = e^{2\pi i \theta} vu)$.

<u>Observe</u> : if $\theta = 0$, then $A_\theta \cong C(\mathbb{T}^2)$ (see Ex (2))

<u>Exer*</u> If $\theta \in \mathbb{R} \setminus \mathbb{Q}$, then $A_\theta \cong$ the C^*-subalg of $\mathcal{B}(L^2(\mathbb{T}))$ gen. by U, V, where

$(Uf)(z) = zf(z)$; $(Vf)(z) = f(e^{-2\pi i \theta} z)$.

(7) $A, B = C^*$-alg $C^*(A \otimes B) = A \otimes_{max} B$.

(8) G = a (discrete) group.

$\mathbb{C}G$ = the group alg of G ; $x^* = x^{-1}$ $(x \in G)$.

<u>Def</u> The (full) <u>group C^*-alg</u> of G is

$$C^*(G) = C^*(\mathbb{C}G).$$

Exer. $C^*(G) = C^*(\ell^1(G))$.

$$\begin{tikzcd} C^*(G) \arrow[rr] & & C^*_r(G) \\ & \mathbb{C}G \arrow[ul] \arrow[ur] & \end{tikzcd}$$

surj. $*$-hom

Fact $C^*(G) \to C^*_r(G)$ is an isom $\iff$ G is amen.

Compact quantum groups

G = a semigroup ; $F(G)$ = an alg. of functions on G.

Case 1: G = affine algebraic semigroup;
$F(G) = \mathcal{O}(G)$ (regular functions on G)

For ex: $G = GL(n, \mathbb{C})$; $G = SL(n,\mathbb{C})$; $G = M_n(\mathbb{C})$

Case 2: G = comp. topol. semigroup; $F(G) = C(G)$

In case 1, $\mathcal{O}(G) \otimes \mathcal{O}(G) \cong \mathcal{O}(G \times G)$
$$f \otimes g \mapsto ((x,y) \mapsto f(x) g(y))$$

In case 2, $C(G) \otimes_* C(G) \cong C(G \times G)$

Let $A = \mathcal{O}(G)$, $\circledast = \otimes$ (alg. $\otimes$) in Case 1;
$A = C(G)$, $\circledast = \otimes_*$ in Case 2.

$m: G\times G \to G$

$$A \xrightarrow{\Delta} A\otimes A$$

$$(\Delta f)(x,y) = f(xy)$$

$$\begin{tikzcd} G\times G\times G \arrow[r, "m\times 1"] \arrow[d, "1\times m"'] & G\times G \arrow[d, "m"] \\ G\times G \arrow[r, "m"] & G \end{tikzcd}
\qquad
\begin{tikzcd} A \arrow[r, "\Delta"] \arrow[d, "\Delta"'] \arrow[dr, phantom, "(*)"] & A\otimes A \arrow[d, "\Delta\otimes 1"] \\ A\otimes A \arrow[r, "1\otimes\Delta"'] & A\otimes A\otimes A \end{tikzcd}$$

Def A coalgebra is (A,Δ), A = vec. space, $\Delta: A \to A\otimes A$ linear s.t. $(*)$ commutes.
Δ = comultiplication; $(*)$: Δ is coassociative

Def (A,Δ) = coalg.
If A is a unital alg and Δ is a unital alg. hom, then (A,Δ) is a bialgebra

Def If A is a unital C^*-alg, $\otimes = \otimes_*$, and Δ is a $*$-hom, then (A,Δ) is a unital C^*-bialgebra.

Morphisms of unital C^*-bialg:

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "\Delta"'] & B \arrow[d, "\Delta"] \\ A\otimes_* A \arrow[r, "\varphi\otimes_*\varphi"] & B\otimes_* B \end{tikzcd}$$

unital $*$-hom

Example G = compact semigroup $\Rightarrow$
$\Rightarrow C(G)$ is a unital C^*-bialg.

Example G = affine algebraic semigroup $\Rightarrow$
$\Rightarrow \mathcal{O}(G)$ is a (unital) bialgebra.

Thm 1 (A, Δ) is a comm. unital C^*-bialg;
$G = \operatorname{Max} A$

$$\begin{array}{ccc} \operatorname{Max}(A \otimes_* A) & \xrightarrow{\Delta^*} & \operatorname{Max} A \\ \text{\rotatebox{90}{$\cong$}} & & \| \\ G \times G & \xrightarrow{m} & G \end{array}$$

Then (G, m) is a compact topol semigroup, and $\Gamma_A : A \to C(G)$ is a unital C^*-bialgebra isomorphism.

Proof. Δ is coassoc. $\Rightarrow m$ is assoc. (G-N)
$\Rightarrow (G, m)$ is a semigroup.
Exer. Γ is a C^*-bialg. morphism. $\square$

Observe: G, H = compact semigr;
$\varphi : G \to H$ cont. hom. $\Rightarrow \varphi^\circ : C(H) \to C(G)$ is a unital C^*-bialg. hom.

Cor. We have an equivalence of categories

$$\{\text{Compact semigroups}\}^{op} \underset{\text{Max}}{\overset{C}{\rightleftarrows}} \left\{\begin{matrix}\text{Comm. unital} \\ C^*\text{-bialg.}\end{matrix}\right\}$$

Goal: construct a category $CQG \subset \left\{\begin{matrix}\text{unital} \\ C^*\text{-bialg}\end{matrix}\right\}$

s.t. $\{\text{Comp. groups}\}^{op} \rightleftarrows \{\text{Comm. CQG}\}$.

Hopf algebras: $(A, \Delta, \varepsilon, S)$ $\varepsilon : A \to \mathbb{C}$ counit

$S : A \to A$ antipode. For ex: $A = \mathcal{O}(G)$.

(G = aff. alg group) $\varepsilon f = f(e)$, $(Sf)(x) = f(x^{-1})$

The C^*-version of this yields a very narrow class of "quantum groups". So we need a different approach.

G = a semigroup

Def. $I \subset G$, $I \neq \emptyset$, is an ideal of $G \iff$

$\iff xI \subset I$, $Ix \subset I$ $\forall x \in G$.

Def G is a semigroup with cancellation $\iff$

$x_1 y = x_2 y \implies x_1 = x_2$, and $y x_1 = y x_2 \implies x_1 = x_2$.

Thm2 G = compact top. semigroup with cancellation $\implies G$ is a topol. group

Proof. Take $x \in G$; H = closed subsemigroup of G gener. by x.

Let $I = \bigcap \{\text{closed ideals of } H\}$.

Compactness $\Rightarrow I \neq \emptyset$ (exer.)

$\forall\, y \in I$ we have $yI = I$

$\Rightarrow \exists\, e \in I$ s.t. $ye = y \Rightarrow \forall\, z \in G \quad ez = z$

$\Rightarrow \forall\, w \in G \quad we = w \Rightarrow e$ is an identity of G.

We have $xe = x$; $e \in I$, $I \subset H$ is an ideal

$\Rightarrow x \in I \Rightarrow xI = I \Rightarrow \exists\, y \in I$ s.t. $xy = e$

$\Rightarrow yx = e \Rightarrow x$ is invertible $\Rightarrow G$ is a group.

The map $G \times G \to G \times G$, $(x,y) \mapsto (x, xy)$,

is a cont. bijection $\Rightarrow$ a homeo. $\Rightarrow$

$\Rightarrow (x,y) \mapsto (x, x^{-1}y)$ is cont. $\Rightarrow x \mapsto x^{-1}$ is cont

$\Rightarrow G$ is a top. group $\quad \square$

<u>Notation.</u> A = an alg, $S_1, S_2 \subset A$ subsets.

$S_1 S_2 = \operatorname{span}\{ab : a \in S_1,\ b \in S_2\}$.

<u>Thm 3</u> $\quad G$ = comp. semigroup, $A = C(G)$;

$\Delta : A \to A \otimes_* A$, $\quad (\Delta f)(x,y) = f(xy)$ TFAE:

(1) G is a top. group;

(2) $\Delta(A)(1 \otimes A)$ and $\Delta(A)(A \otimes 1)$ are dense in $A \otimes_* A$.

Proof. (1) $\Rightarrow$ (2)

$\Delta(A)(1\otimes A) = \mathrm{span}\{(x,y) \mapsto f(xy)g(y) \mid f,g\in A\}$

$\Delta(A)(A\otimes 1) = \mathrm{span}\{(x,y) \mapsto f(xy)g(x) \mid f,g\in A\}$

These subspaces are $*$-subalg. satisfying the cond's of the Stone-Weier. thm.

(2) $\Rightarrow$ (1) Suppose $x_1 y = x_2 y$ $\Rightarrow$

$\Rightarrow \forall\, h\in \Delta(A)(1\otimes A)$ we have $h(x_1,y) = h(x_2,y)$

$\Rightarrow$ the same holds $\forall\, h \in A\otimes_* A = C(G\times G)$

$\Rightarrow x_1 = x_2$. Similarly: $yx_1 = yx_2 \Rightarrow x_1 = x_2$.

Thm 2 $\Rightarrow$ G is a top. group $\square$.

Def A <u>compact quantum group</u> is a unital C^*-bialgebra (A,Δ) s.t. $\Delta(A)(1\otimes A)$ and $\Delta(A)(A\otimes 1)$ are dense in $A\otimes_* A$.

(S.L. Woronowicz 2000)

Example G = comp. group $\Rightarrow C(G)$ is a CQG.

Cor. We have an equiv. of categories

$$\{\text{comp groups}\}^{op} \rightleftarrows \{\text{Comm. CQG}\}$$