Recall: $A, B = C^*$-alg. $\quad u \in A \otimes B$

$$\|u\|_{max} = \sup\{\|\pi(u)\| : \pi = *\text{-rep. of } A \otimes B\}.$$

Thm (1) $\|u\|_{max} < \infty \ \forall u \in A \otimes B$.

(so $\| \, \|_{max}$ is a C^*-norm on $A \otimes B$).

(2) $\forall$ C^*-alg C $\forall$ $*$-hom $\varphi : A \otimes B \to C$ we have $\|\varphi(u)\| \le \|u\|_{max}$ $(u \in A \otimes B)$

(3) $\| \cdot \|_{max}$ is the largest C^*-norm on $A \otimes B$.

(4) $\|a \otimes b\|_{max} = \|a\| \|b\|$ $(a \in A, b \in B)$.

Def $A \otimes_{max} B = \text{completion}(A \otimes B, \| \cdot \|_{max})$.

Cor 1 $\forall$ C^*-alg C each $*$-hom $\varphi : A \otimes B \to C$ uniquely extends to a $*$-hom $A \otimes_{max} B \to C$.

Cor. 2 Suppose C is a C^*-alg, $\varphi : A \to C$ and $\psi : B \to C$ are $*$-homs, $[\varphi, \psi] = 0$. Then $\exists$ a unique $*$-hom $\pi : A \otimes_{max} B \to C$ s.t. $\pi(a \otimes b) = \varphi(a)\psi(b)$ $(a \in A, b \in B)$

Proof Apply Cor 1 to $\varphi \times \psi : A \otimes B \to C$, $a \otimes b \mapsto \varphi(a)\psi(b)$. $\square$

Exer. Not every π comes from (φ, ψ).

Cor. 3. $\left\{\begin{matrix}\text{Nondeg. } *\text{-reps} \\ \text{of } A \otimes_{max} B \text{ on } H\end{matrix}\right\} \underset{\leftarrow}{\overset{1-1}{\rightarrow}} \left\{\begin{matrix}(\pi_1, \pi_2) & \pi_1, \pi_2 \text{ are nondeg} \\ *\text{-reps of } A, B \text{ on } H, \\ [\pi_1, \pi_2] = 0\end{matrix}\right\}$

Lemma. $A, B = C^*$-alg, $\varphi: A \to B$ $*$-hom.
Then $\varphi(A)$ is closed in B, and $\|\varphi\| = 1$ (if $\varphi \neq 0$)

Proof:
$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "q"'] & B \\ A/\operatorname{Ker}\varphi \arrow[ur, "\hat{\varphi}"'] & \end{tikzcd}$$

$\hat{\varphi}$ is inj. $\Rightarrow$
$\Rightarrow$ isometric $\Rightarrow$
$\Rightarrow$ $\hat{\varphi}(A/\operatorname{Ker}\varphi)$ is closed
$\hat{\varphi}(A/\operatorname{Ker}\varphi) = \varphi(A)$

$\|\varphi\| = \|q\| = 1$ (unless $\varphi = 0$) $\square$

Cor. 4. The identity map of $A \otimes B$ extends to a surj. $*$-hom $A \otimes_{max} B \to A \otimes_{min} B$.

Def A C^*-alg A is <u>nuclear</u> $\Leftrightarrow$ $\forall$ C^*-alg B
$\|\cdot\|_{min} = \|\cdot\|_{max}$ on $A \otimes B$ (<u>ядерная</u>)
$\Leftrightarrow$ $A \otimes_{max} B \to A \otimes_{min} B$ is an isomorphism.

Example M_n is nuclear
(because $\exists$ a unique C^*-norm on $M_n \otimes B$).

Example/exer. Each fin-dim C^*-alg is isom. to $M_{n_1} \oplus \dots \oplus M_{n_k}$ $\Rightarrow$ it is nuclear.

Thm $A = C^*$-alg; $\mathcal{B}$ = a directed family of C^*-subalg of A
(that is, $\forall\, B_1, B_2 \in \mathcal{B}\ \exists\, B_3 \in \mathcal{B}$ s.t. $B_1 \cup B_2 \subset B_3$).
Suppose $\cup\mathcal{B}$ is dense in A, and each $B \in \mathcal{B}$ is nuclear $\Rightarrow$ A is nuclear.

Proof $C = C^*$-alg; $A_0 = \cup\mathcal{B}$.
A_0 is a dense $*$-subalg of A.
$A_0 \otimes C = \cup\{B \otimes C : B \in \mathcal{B}\} \Rightarrow$
$\Rightarrow \|\cdot\|_{max} = \|\cdot\|_{min}$ on $A_0 \otimes C$.
$A_0 \otimes C$ is dense in $(A \otimes C, \|\cdot\|_{max})$ (exer.);
$\|\cdot\|_{min} \le \|\cdot\|_{max} \Rightarrow \|\cdot\|_{min}$ is cont. wrt $\|\cdot\|_{max}$.
$\Rightarrow \|\cdot\|_{min} = \|\cdot\|_{max}$ on $A \otimes C$. $\square$

Cor. $\mathcal{K}(H)$ is nuclear.

Proof $L \subset H$ fin-dim. vec. subspace;
P_L = the orth. proj. onto L.
$A_L = \{P_L T P_L : T \in \mathcal{K}(H)\}$.
A_L is a $*$-subalg of $\mathcal{K}(H)$, $A_L \cong M_{\dim L}$ (exer.).
$\mathcal{B} = \{A_L : L \subset H, \dim L < \infty\}$ satisfies the cond's of Thm. (exer.) $\square$

Facts. (1) Each comm. C^*-alg is nuclear (Takesaki 1964).

(2) $C^*_r(F_2)$ is not nuclear (Takesaki 1964)

$\mathcal{B}(H)$ is not nuclear (Szankowski 1981)

Remarks on locally convex tensor products and nuclear spaces

$\hat{\otimes}_\pi$ proj $\qquad$ $\hat{\otimes}_\varepsilon$ inj.

$$X \hat{\otimes}_\pi Y = (X \otimes Y, \| \; \|_\pi)^\sim$$

$$X \hat{\otimes}_\varepsilon Y = (X \otimes Y, \| \; \|_\varepsilon)^\sim$$

$$\| \cdot \|_\varepsilon \le \| \cdot \|_\pi$$

$$X \hat{\otimes}_\pi Y \longrightarrow X \hat{\otimes}_\varepsilon Y. \quad (*)$$

Def. X = lcs.

X is <u>nuclear</u> $\iff$ $\forall$ lcs Y $(*)$ is a topol. isom.

$C^\infty(M)$, $\mathcal{O}(M)$; $\mathcal{S}(\mathbb{R}^n)$; $\mathcal{E}'(\mathbb{R}^n)$; $\mathcal{S}'(\mathbb{R}^n)$; ...

are nuclear.

$\{\text{normed}\} \cap \{\text{nuclear}\} = \{\text{fin-dim}\}$.

Hence inf-dim. nuclear C^*-algebras are never nuclear as locally convex spaces.

$A, B = C^*$-alg. $\qquad \| \cdot \|_\varepsilon \le \| \cdot \|_{min} \le \| \cdot \|_{max} \le \| \cdot \|_\pi$.

$\| \cdot \|_\varepsilon = \| \cdot \|_{min}$ if A or B is commutative.

Otherwise $\| \cdot \|_\varepsilon$ is not a C^*-norm.

$\| \cdot \|_\pi$ is (almost) never a C^*-norm.

C^*-envelopes

$A = *$-algebra

<u>Def.</u> A <u>C^*-envelope</u> of A is $(C^*(A), \theta)$ where $C^*(A)$ is a C^*-alg, $\theta : A \to C^*(A)$ $*$-hom. s.t.

$$\begin{tikzcd} C^*(A) \arrow[r, "\psi"] & B \\ A \arrow[u, "\theta"] \arrow[ur, "\varphi"'] & \end{tikzcd}$$

$\forall$ C^*-alg B $\forall$ $*$-hom $\varphi : A \to B$ $\exists$ a unique $*$-hom $\psi : C^*(A) \to B$ s.t. the diagram commutes.

<u>Observe</u>:

$$\begin{tikzcd} C_1^*(A) \arrow[rr, "\psi", "\sim"'] & & C_2^*(A) \\ & A \arrow[ul, "\theta_1"] \arrow[ur, "\theta_2"'] & \end{tikzcd}$$

$\exists$ a unique $*$-isom ψ

<u>Def</u> A <u>C^*-seminorm</u> on A is a seminorm p on A s.t.

(1) $p(ab) \le p(a)p(b)$;

(2) $p(a^*) = p(a)$; $\qquad (a, b \in A)$

(3) $p(a^*a) = p(a)^2$

<u>Notation</u> $p =$ a C^*-semi. on A ; $I_p = p^{-1}(0)$

I_p is a 2-sided $*$-ideal of A.

$A_p^0 = (A/I_p, \|\cdot\|_p)$ $\qquad \|a + I_p\|_p = p(a)$. $(a \in A)$ is a C^*-norm on A_p^0.

$A_p =$ completion $(A_p^0, \|\cdot\|_p)$. $\qquad A_p$ is a C^*-alg.

$\pi_p : A \to A_p, \quad a \mapsto a + I_p. \qquad *$-hom.

$\forall a \in A$ let $\|a\|^* = \sup\{p(a) : p = \text{a } C^*\text{-semi. on } A\}$
$\in [0, +\infty]$.

<u>Prop.</u> (1) $\|a\|^* = \sup\{\|\pi(a)\| : \pi \text{ is a } *\text{-rep of } A\}$.

(2) $C^*(A)$ exists $\iff \|a\|^* < \infty \ \forall a \in A$.

(3) Suppose $\|a\|^* < \infty \ \forall a \in A$. Then $p = \|\cdot\|^*$ is the largest C^*-semi on A, and (A_p, π_p) is a C^*-env of A.

<u>Proof</u> (1) $(\geq)$ clear because $a \mapsto \|\pi(a)\|$ is a C^*-semi.

$(\leq)$ p = a C^*-semi. on A. $\quad \tau : A_p \hookrightarrow \mathcal{B}(H)$.

$\forall a \in A \quad p(a) = \|\pi_p(a)\| = \|(\tau\pi_p)(a)\|$

(2)$(\Rightarrow)$ Suppose $C^*(A)$ exists.

$$\begin{tikzcd} C^*(A) \arrow[r, "\psi_p"] & A_p \\ A \arrow[u, "\theta"] \arrow[ur, "\pi_p"'] & \end{tikzcd}$$

$p(a) = \|\pi_p(a)\| = \|(\psi_p\theta)(a)\| \leq$
$\leq \|\theta(a)\|$.

$\Rightarrow \|a\|^* \leq \|\theta(a)\| < \infty$.

(2)$(\Leftarrow)$ and (3) $\quad B = C^*\text{-alg}, \ \varphi : A \to B \ *$-hom

$q(a) = \|\varphi(a)\|$ is a C^*-semi on $A \Rightarrow q \leq p$

$\Rightarrow \varphi(I_p) = 0 \Rightarrow \exists$ a unique $*$-hom φ_0

making the diag(below) commute.

$$\begin{tikzcd}
A_p \arrow[dr, "\psi"] & \\
A_p^0 \arrow[u, hook] \arrow[r, "\psi_0"] & B \\
A \arrow[u] \arrow[ur, "\varphi"'] \arrow[uu, bend left=60, "\pi_p"] &
\end{tikzcd}$$

$\|\psi_0(a+I_p)\| = \|\varphi(a)\| \le p(a)$

$\Rightarrow \psi_0$ is bdd $\Rightarrow \psi_0$ uniquely extends to $\psi : A_p \to B$. $\square$

Prop. A = $*$-alg generated (as a $*$-alg) by $S \subset A$. Suppose $\forall a \in S\ \exists C > 0$ s.t. $\forall$ C^*-semi p on A we have $p(a) \le C$. Then $C^*(A)$ exists.

Proof. Each $a \in A$ has the form $a = \sum a_{i_1} \dots a_{i_n}$ (fin. sum); $a_{i_j} \in S \cup S^*$. $\exists C > 0$ s.t. $\forall p$ $p(a_{i_j}) \le C \Rightarrow \exists D > 0$ s.t. $p(a) \le D$, where D does not depend on $p \Rightarrow \|a\|^* < \infty$. $\square$

Prop. A = unital $*$-alg, $u \in M_n(A)$ unitary. Then $\forall$ C^*-semi $\|\cdot\|$ on A $\|u_{ij}\| \le 1$ $\forall i, j$.

Proof. We may assume that A is a C^*-alg (consider A_p, where $p = \|\cdot\|$)

$A \hookrightarrow \mathcal{B}(H)$. $\Rightarrow$ we may assume $A = \mathcal{B}(H)$.

$u \in M_n(\mathcal{B}(H)) = \mathcal{B}(H^n)$

$u_{ij} = p_i u e_j$, where p_i is the proj $H^n \to H$

$e_j : H \hookrightarrow H^n$, $x \mapsto (0 \dots 0 x 0 \dots)$

$\|u_{ij}\| \leq \|u\| = 1 \quad \forall i, j. \quad \square$

<u>Cor.</u> A = unital $*$-alg generated by the entries of a family of unitary matrices over A. Then $C^*(A)$ exists.

<u>Terminology</u> I = a set;
$F = \mathbb{C}\langle x_i, y_i \mid i \in I \rangle$ free alg.

F is a $*$-alg wrt the invol uniquely det'd by $x_i^* = y_i \ \forall i \in I$.
$\{p_j : j \in J\}$ = family of elements of F;
K = 2-sided $*$-ideal of F gen. by $\{p_j : j \in J\}$ (as a $*$-ideal).

<u>Def</u> The <u>universal unital C^*-alg</u> gener. by $\{x_i : i \in I\}$ with relations $\{p_j : j \in J\}$ is
$C^*(x_i \mid p_j) \overset{\text{def}}{=} C^*(F/K)$ (if it exists)

<u>Univ. property</u>: $\forall$ C^*-alg A, $\forall$ family $\{a_i : i \in I\}$ of elements of A s.t. $p_j(a_i, a_i^*) = 0 \ \forall j \in J$
$\exists$ a unique unital $*$-hom $\alpha : C^*(x_i \mid p_j) \to A$ s.t. $\alpha(x_i) = a_i \ \forall i \in I$.

<u>Notation</u> $C^*_{\text{comm}}(x_i \mid p_j) = C^*\left(x_i \mid p_j ;\ \begin{matrix} x_i x_k - x_k x_i \\ x_i x_k^* - x_k^* x_i \end{matrix}\right)$