Recall: $A, B = C^*$-alg.

$\omega_A : A \to \mathcal{B}(H_A)$, $\omega_B : B \to \mathcal{B}(H_B)$ univ. rep.

$\omega_A \dot{\otimes} \omega_B : A \odot B \to \mathcal{B}(H_A \dot{\otimes} H_B)$

$\forall\ u \in A \odot B \quad \|u\|_* = \|(\omega_A \dot{\otimes} \omega_B)(u)\|$

$$\begin{array}{ccc} A \otimes_* B & = \text{completion } (A \odot B, & \|\cdot\|_*) \\ \| & & \| \\ A \otimes_{min} B & & \|\cdot\|_{min} \end{array}$$

Thm $\|u\|_* = \sup \left\{ \|(\pi \dot{\otimes} \tau)(u)\| : \begin{array}{l} \pi = *\text{-rep of } A \\ \tau = *\text{-rep of } B \end{array} \right\}$.

Cor. $A_1 \xrightarrow{\varphi} B_1$, $A_2 \xrightarrow{\psi} B_2$ $*$-homs

$\Rightarrow \exists$ a unique $*$-hom $\varphi \otimes_* \psi : A_1 \otimes_* A_2 \to B_1 \otimes_* B_2$

s.t. $a_1 \otimes a_2 \mapsto \varphi(a_1) \otimes \psi(a_2)$

Thm $\pi : A \to \mathcal{B}(H)$, $\tau : B \to \mathcal{B}(K)$ faithful $*$-reps

Then $\|u\|_* = \|(\pi \dot{\otimes} \tau)(u)\| \quad (u \in A \odot B)$.

Lemma $FD(H) = \{L \subset H : \dim L < \infty\}$.

$\forall\ L \in FD(H)$ P_L = orth proj onto L.

Then $\forall\ T \in \mathcal{B}(H \dot{\otimes} K)$

$$\|T\| = \sup_{L \in FD(H)} \|(P_L \dot{\otimes} 1) T (P_L \dot{\otimes} 1)\|$$

Proof. $T \in \mathcal{B}(H \dot{\otimes} K)$; $\forall \varepsilon > 0$ choose $u \in H \otimes K$ s.t. $\|u\| = 1$, $\|T\| \leq \|Tu\| + \varepsilon$.

$u = \sum_{i=1}^{n} x_i \otimes y_i$.

Observe: $\forall\, v \in H \dot{\otimes} K$ $(P_L \dot{\otimes} 1)v \to v$ $(L \in FD(H))$

(because this is true for $v = x \otimes y$, and $(P_L \otimes 1)$ is bdd).

$\Rightarrow \exists\, L \in FD(H)$ s.t. $x_1, \ldots, x_n \in L$ and s.t.

$\| Tu - (P_L \dot{\otimes} 1) Tu \| < \varepsilon$.

$$\Rightarrow \|T\| \leq \|Tu\| + \varepsilon \leq \|(P_L \dot{\otimes} 1)(Tu)\| + 2\varepsilon =$$
$$= \|(P_L \otimes 1) T \underbrace{(P_L \dot{\otimes} 1) u}_{u} \| + 2\varepsilon. \quad \square$$

Proof of Thm Let $\pi : A \to \mathcal{B}(H)$, $\pi' : A \to \mathcal{B}(H')$, $\tau : B \to \mathcal{B}(K)$, $\tau' : B \to \mathcal{B}(K')$ be faithful $*$-reps.

Let $\|u\|_{\pi,\tau} = \|(\pi \dot{\otimes} \tau)(u)\|$ $\quad (u \in A \otimes B)$

We want : $\|u\|_{\pi,\tau} = \|u\|_{\pi',\tau'}$.

We may assume that $\pi = \pi'$!

$\forall\, L \in FD(H)$ let $\varphi_L : A \to \mathcal{B}(L)$, $\varphi_L(a) = P_L \pi(a) P_L$.

We have

$$(P_L \dot{\otimes} 1)\big((\pi \dot{\otimes} \tau)(u)\big)(P_L \dot{\otimes} 1) = (1_{\mathcal{B}(L)} \dot{\otimes} \tau)\big((\varphi_L \otimes 1_B)(u)\big)$$

$$\begin{tikzcd}
 & \mathcal{B}(L)\otimes B \arrow[dl, "1\dot{\otimes}\tau"'] \arrow[dr, "1\dot{\otimes}\tau'"] & \\
\mathcal{B}(L\dot{\otimes}K) & & \mathcal{B}(L\dot{\otimes}K')
\end{tikzcd}$$

faithful $*$-reps

$\Rightarrow \forall\ v\in\mathcal{B}(L)\otimes B \quad \|(1\dot{\otimes}\tau)(v)\| = \|(1\dot{\otimes}\tau')(v)\|$

(because $\mathcal{B}(L)\otimes B = M_n\otimes B$, and $\exists$ a <u>unique</u> C^*-norm on $M_n\otimes B$)

$$\Rightarrow \|u\|_{\pi,\tau} \overset{L}{=} \sup_L \|(1\dot{\otimes}\tau)((\varphi\otimes 1)(u))\| =$$

$$= \sup_L \|(1\dot{\otimes}\tau')((\varphi\otimes 1)(u))\| \overset{L}{=} \|u\|_{\pi,\tau'}. \qquad \square$$

<u>Cor 1</u> $A\subset\mathcal{B}(H)$, $B\subset\mathcal{B}(K)$ C^*-subalg

$\Rightarrow A\otimes_* B$ = the closure of $A\otimes B$ in $\mathcal{B}(H\dot{\otimes}K)$.

<u>Cor. 2</u> $\mathcal{K}(H_1)\otimes_*\mathcal{K}(H_2)\cong\mathcal{K}(H_1\dot{\otimes}H_2)$.

<u>Notation</u> H = Hilb. sp, $x,y\in H$.

$x\odot y: H\to H$, $(x\odot y)(z) = \langle z|y\rangle x$.

$\mathrm{span}\{x\odot y \mid x,y\in H\} = \{$bdd. fin. rank operators$\}$

is dense in $\mathcal{K}(H)$ (fact).

<u>Proof of Cor 2</u>

$$\mathcal{K}(H_1)\otimes_*\mathcal{K}(H_2) = \overline{\mathcal{K}(H_1)\otimes\mathcal{K}(H_2)}^{\,\mathcal{B}(H_1\dot{\otimes}H_2)}$$

$$= \overline{\operatorname{span}\{x_1 \odot y_1 \mid x_1, y_1 \in H_1\} \otimes \operatorname{span}\{x_2 \odot y_2 \mid x_2, y_2 \in H_2\}}$$

$$= \overline{\operatorname{span}\{(x_1 \otimes x_2) \odot (y_1 \otimes y_2) \mid x_1, y_1 \in H_1,\ x_2, y_2 \in H_2\}}$$

$$= \overline{\operatorname{span}\{\xi \odot \eta \mid \xi, \eta \in H_1 \dot{\otimes} H_2\}} = \mathcal{K}(H_1 \dot{\otimes} H_2) \qquad \square$$

Exer. $\mathcal{B}(H_1) \otimes_* \mathcal{B}(H_2) \xhookrightarrow[\text{isometr.}]{} \mathcal{B}(H_1 \dot{\otimes} H_2)$.

Is it surj ?

Notation X = loc. comp. Hausd. top. space,

E = Ban. space

$C_0(X, E) = \{\text{cont. } f : X \to E \mid x \mapsto \|f(x)\| \text{ vanishes at } \infty\}$

Exer. (1) $C_0(X, E)$ is a Banach space w.r.t.

$$\|f\| = \sup_{x \in X} \|f(x)\|.$$

(2) $A = C^*$-alg $\Rightarrow$ $C_0(X, A)$ is a C^*-alg

w.r.t. $f^*(x) = f(x)^* \quad (x \in X)$

(3) $C_0(X) \otimes E \xrightarrow{\varphi} C_0(X, E)$,

$\varphi(f \otimes v)(x) = f(x)v \quad (x \in X)$

Then φ has dense image.

(Hint: use partitions of unity).

Thm X = loc. comp. Hausd. top. space, $A = C^*$-alg. Then $\exists$ an isometric $*$-isom $\varphi : C_0(X) \otimes_* A \xrightarrow{\sim} C_0(X, A)$ uniquely det'd by $\varphi(f \otimes a)(x) = f(x)a \quad (x \in X)$

Proof $\pi : A \to \mathcal{B}(H)$ faithful $*$-rep;

$M : C_0(X) \to \mathcal{B}(\ell^2(X))$, $M(f) = M_f$,

$$M_f(g) = fg \quad (f \in C_0(X),\ g \in \ell^2(X)).$$

M is a faithful $*$-rep.

$$\begin{tikzcd} C_0(X) \otimes A \arrow[r, "M \dot{\otimes} \pi"] \arrow[d, "\varphi"'] & \mathcal{B}(\ell^2(X) \dot{\otimes} H) \arrow[d, "Ad(u)"] \\ C_0(X, A) \arrow[r, "M_\pi"] & \mathcal{B}(\ell^2(X, H)) \end{tikzcd}$$

Here $u : \ell^2(X) \dot{\otimes} H \xrightarrow{\sim} \ell^2(X, H)$, unitary isom

$$u(g \otimes h)(x) = g(x)h.$$

$Ad(u)T = uTu^{-1}$ isometric $*$-isom.

$(M_\pi f)(g)(x) = \pi(f(x))g(x) \quad (f \in C_0(X, A),\ g \in \ell^2(X, H)).$

faithful $*$-rep (exer)

$\Rightarrow$ isometric. The diag is comm. (exer)

$\Rightarrow \varphi : (C_0(X)\otimes A, \|\cdot\|_*) \longrightarrow C_0(X,A)$

is an isometric $*$-hom with dense image.

$\Rightarrow$ φ extends to an isometric $*$-isom

$$C_0(X)\otimes_* A \xrightarrow{\sim} C_0(X,A) \qquad \square$$

<u>Cor.</u> X, Y = loc. comp. Hausd. top. spaces $\Rightarrow$

$\Rightarrow$ $\exists$ an isometric $*$-isom

$$C_0(X)\otimes_* C_0(Y) \underset{\varphi}{\xrightarrow{\sim}} C_0(X\times Y),$$

$$\varphi(f\otimes g)(x,y) = f(x)g(y) \quad (x\in X, y\in Y)$$

<u>Proof</u> Apply Thm to $A = C_0(Y)$ and use the isom. $C_0(X, C_0(Y)) \cong C_0(X\times Y)$ (exer) $\square$

The maximal tensor product

A, B = C^*-alg. $\qquad u\in A\otimes B$

<u>Recall</u>: $\|u\|_{min} = \sup\left\{ \|(\pi_1 \dot\otimes \pi_2)(u)\| : \begin{matrix} \pi_1 = *\text{-rep of } A \\ \pi_2 = *\text{-rep of } B \end{matrix} \right\}$

<u>Def</u> $\|u\|_{max} = \sup\{\|\pi(u)\| : \pi = *\text{-rep of } A\otimes B\} \in [0,+\infty]$.

Clearly, $\|\cdot\|_{min} \le \|\cdot\|_{max}$.

Why is $\|u\|_{max} < \infty$?

Notation A, B, C algebras,
$\varphi : A \to C$, $\psi : B \to C$ homs.
$\varphi \times \psi : A \otimes B \to C$, $a \otimes b \mapsto \varphi(a)\psi(b)$.

Def φ and ψ commute ($[\varphi, \psi] = 0$) $\iff$
$\iff [\varphi(a), \psi(b)] = 0 \ \forall a \in A, b \in B$.

Prop. Suppose $[\varphi, \psi] = 0$. Then
(1) $\varphi \times \psi$ is an alg. hom;
(2) if A, B, C are $*$-alg, φ, ψ are $*$-homs, then $\varphi \times \psi$ is a $*$-hom.

Proof: exer.

Observation Suppose $\pi : A \otimes B \to C$ is an alg. hom, A, B are unital.
Define $\varphi : A \to C$, $\psi : B \to C$ by $\varphi(a) = \pi(a \otimes 1)$, $\psi(b) = \pi(1 \otimes b)$ $\implies$ φ, ψ are alg. homs, $[\varphi, \psi] = 0$, and $\pi = \varphi \times \psi$.

Thm $A, B = C^*$-alg, $\pi : A \otimes B \to \mathcal{B}(H)$ nondeg. $*$-rep. Then $\exists$ a unique pair (π_A, π_B) of nondeg $*$-reps of A, B s.t. $[\pi_A, \pi_B] = 0$ and $\pi = \pi_A \times \pi_B$.

<u>Thm</u> $A, B = C^*$-alg. Then

(1) $\|u\|_{max} < \infty \ \forall u \in A \otimes B$.

(so $\|\cdot\|_{max}$ is a C^*-norm on $A \otimes B$).

(2) $\forall$ C^*-alg C $\forall$ $*$-hom $\varphi : A \otimes B \to C$ we have $\|\varphi(u)\| \le \|u\|_{max}$ $(u \in A \otimes B)$.

(3) $\|\ \|_{max}$ is the largest C^*-norm on $A \otimes B$.

(4) $\|a \otimes b\|_{max} = \|a\| \|b\|$ $(a \in A, b \in B)$.

<u>Proof</u> (1) We want: $\forall u \in A \otimes B \ \exists C > 0$ s.t. $\|\pi(u)\| \le C$ $\forall$ $*$-rep $\pi : A \otimes B \to \mathcal{B}(H)$.

We may assume that π is nondeg.

$\Rightarrow \pi = \pi_A \times \pi_B$ (see the prev. thm.)

$$u = \sum_{i=1}^n a_i \otimes b_i \implies$$

$$\Rightarrow \|\pi(u)\| = \Big\| \sum_i \pi_A(a_i) \pi_B(b_i) \Big\| \le \sum \|a_i\| \|b_i\| = C.$$

(2) Clear if $C = \mathcal{B}(H)$.

In the gen. case apply 2nd GN to C.

(3) Suppose $\|\ \|$ is a C^*-norm on $A \otimes B$

C = completion of $A \otimes B$ wrt $\|\cdot\|$.

Apply (2) to $A \otimes B \hookrightarrow C$.

(4) π = nondeg. $*$-rep of $A\otimes B$, $\pi = \pi_A \times \pi_B$

$\|\pi(a\otimes b)\| = \|\pi_A(a)\pi_B(b)\| \leq \|a\|\|b\| \implies$

$\implies \|a\otimes b\|_{max} \leq \|a\|\|b\| = \|a\otimes b\|_* \leq \|a\otimes b\|_{max}.$

$\square$

<u>Def</u> The <u>maximal C^*-tensor product</u> of A and B is the completion of $A\otimes B$ wrt $\|\cdot\|_{max}$.

<u>Notation</u>: $A\otimes_{max} B$.