Gap $A = C^*$-alg, $f \in A^*$ pos; $(e_\lambda) =$ a.i. in A $\Rightarrow$
$\Rightarrow (e_\lambda^2)$ is an a.i. $\Rightarrow f(e_\lambda) \to \|f\|$, $f(e_\lambda^2) \to \|f\|$
$\Rightarrow \Lambda_{f_+}(e_\lambda) \to \Lambda_{f_+}(1_+)$ $\Rightarrow H_f \xrightarrow{\sim} H_{f_+} \Rightarrow \pi_f$ is cyclic

Problem: (e_λ^2) is not necess. monotone!

Correction:

Lemma $A = C^*$-alg, $f \in A^*$ pos., $(e_\lambda) =$ a.i. in A s.t. $e_\lambda \geq 0$, $\|e_\lambda\| \leq 1$, but is not necess. monot.
$\Rightarrow \exists \lim f(e_\lambda) = \|f\|$.

Proof. We may assume $\|f\| = 1$.
$\forall a \in A$, $\|a\| \leq 1$, we have $0 \leq f(a^*a) \leq 1$.
$|f(e_\lambda a)|^2 \leq f(e_\lambda^2) f(a^*a) \leq f(e_\lambda)$ because $e_\lambda^2 \leq e_\lambda$.
$\Rightarrow |f(a)|^2 \leq \underline{\lim} f(e_\lambda) \leq \overline{\lim} f(e_\lambda) \leq \sup f(e_\lambda) \leq 1$.
Take $\sup\limits_{\|a\| \leq 1}$: $\Rightarrow \exists \lim f(e_\lambda) = 1$. $\square$

Thanks Kostya!

Tensor products of C^*-algebras

<u>We know:</u>

① A, B $*$-alg $\Rightarrow A\otimes B$ is a $*$-alg;
$(a_1\otimes b_1)(a_2\otimes b_2) = a_1a_2\otimes b_1b_2$;
$(a\otimes b)^* = a^*\otimes b^*$.

② $A = C^*$-alg; $M_n = \{n\times n\text{-matrices with entries in } \mathbb{C}\}$
$\Rightarrow M_n\otimes A$ is a C^*-alg.
Moreover: $\exists$ a unique C^*-norm on $M_n\otimes A$.

<u>Prop</u> H_1, H_2 = inn. prod. spaces. $\Rightarrow$
$\exists$ a unique inn. product on $H_1\otimes H_2$ satisf.

$$\langle x_1\otimes x_2 | y_1\otimes y_2\rangle = \langle x_1|y_1\rangle\langle x_2|y_2\rangle \qquad (*)$$

<u>Proof</u> Observe: $\forall$ vec. sp E, F
$$\{\text{sesquilin. } E\times F\to\mathbb{C}\} \cong \{\text{bilin } E\times F^c\to\mathbb{C}\}$$
$$\cong \operatorname{Hom}_{\mathbb{C}}(E\otimes F^c, \mathbb{C})$$

$$(H_1\otimes H_2)\otimes(H_1\otimes H_2)^c \cong H_1\otimes H_2\otimes H_1^c\otimes H_2^c \cong$$
$$\cong (H_1\otimes H_1^c)\otimes(H_2\otimes H_2^c) \xrightarrow[\langle\cdot|\cdot\rangle_{H_1}\otimes\langle\cdot|\cdot\rangle_{H_2}]{} \mathbb{C}\otimes\mathbb{C}\cong\mathbb{C}.$$

$\Rightarrow \exists$ a unique sesq. form $\langle\cdot|\cdot\rangle$ on $H_1\otimes H_2$ satisfying $(*)$.

$\forall u \in H_1 \otimes H_2 \quad u = \sum_{i=1}^{n} y_i \otimes z_i$;

let $\{e_1, \dots, e_m\}$ be an ON basis in $\mathrm{span}\{z_1, \dots, z_n\}$.

$\Rightarrow u = \sum_{i=1}^{m} x_i \otimes e_i \quad (x_i \in H_1)$

$\Rightarrow \langle u | u \rangle = \sum_{i=1}^{m} \|x_i\|^2 > 0$ if $u \neq 0$. $\square$

<u>Observe</u>: $\|x \otimes y\| = \|x\| \|y\| \quad (x \in H_1, y \in H_2)$

(that is, the norm on $H_1 \otimes H_2$ is a <u>crossnorm</u>)

<u>Def</u> H_1, H_2 = Hilb. spaces. Their <u>Hilbert tensor product</u> is the completion of $H_1 \otimes H_2$ w.r.t. $\langle \cdot | \cdot \rangle$ given by $(*)$.

<u>Notation</u>: $H_1 \dot{\otimes} H_2$

<u>Exer</u> $(H_1 \otimes H_2, \langle \cdot | \cdot \rangle)$ is complete $\Leftrightarrow$

$\Leftrightarrow H_1$ or H_2 is fin-dim.

<u>Exer</u> $(e_i)_{i \in I}$ = ONB in H_1 ; $(f_j)_{j \in J}$ = ONB in H_2

$\Rightarrow (e_i \otimes f_j)_{(i,j) \in I \times J}$ is an ONB of $H_1 \dot{\otimes} H_2$.

<u>Examples / exer</u>

(1) $H \dot{\otimes} \mathbb{C}^n \cong H \dot{\oplus} \dots \dot{\oplus} H$

(2) $H \dot{\otimes} \ell^2(I) \cong \dot{\bigoplus}_{i \in I} H \cong \ell^2(I, H)$

(3) $L^2(X,\mu)\dot{\otimes} L^2(Y,\nu) \cong L^2(X\times Y, \mu\times\nu)$.

<u>Prop</u>. H_1, H_2, K_1, K_2 = Hilb. spaces, $S\in\mathscr{B}(H_1,K_1)$, $T\in\mathscr{B}(H_2,K_2)$ $\Rightarrow$ $\exists$ a unique $S\dot{\otimes}T \in \mathscr{B}(H_1\dot{\otimes}H_2, K_1\dot{\otimes}K_2)$ satisfying

$$(S\dot{\otimes}T)(x\otimes y) = Sx\otimes Ty \quad (x\in H_1,\ y\in H_2)\ (**)$$

Moreover, $\|S\dot{\otimes}T\| = \|S\|\,\|T\|$.

<u>Proof</u>. Uniqueness: clear;

$\|S\dot{\otimes}T\| \geqslant \|S\|\,\|T\|$ clear.

Existence: $\exists$ a unique lin. $S\otimes T: H_1\otimes H_2 \to K_1\otimes K_2$ satisfying $(**)$.

Now it suff. to show that $S\otimes T$ is bdd and that $\|S\otimes T\| \leq \|S\|\,\|T\|$.

We may assume that $T = 1_H$ (H = a Hilb. space)

(because $S\otimes T = (S\otimes 1)(1\otimes T)$).

Let $u\in H_1\otimes H$, $u = \sum_{i=1}^n x_i\otimes e_i$ ($e_1,\ldots,e_n$ = ON family)

$\Rightarrow (S\otimes 1)(u) = \sum_{i=1}^n Tx_i\otimes e_i$.

$\Rightarrow \|(S\otimes 1)(u)\|^2 = \sum_{i=1}^n \|Sx_i\|^2 \leqslant \|S\|^2 \sum_{i=1}^n \|x_i\|^2 = \|S\|^2\|u\|^2$

$\Rightarrow S\otimes 1$ is bdd, $\|S\otimes 1\| \leq \|S\|$. $\square$

Prop. $\exists$ an injective lin. map

$$\mathcal{B}(H_1,K_1)\otimes\mathcal{B}(H_2,K_2)\xrightarrow{\alpha}\mathcal{B}(H_1\dot{\otimes}H_2,\ K_1\dot{\otimes}K_2),$$

$$S\otimes T\longmapsto S\dot{\otimes}T.$$

Proof. Let $U\in\mathcal{B}(H_1,K_1)\otimes\mathcal{B}(H_2,K_2)$, $\alpha(U)=0$.

$U=\sum_{i=1}^{n}S_i\otimes T_i$, $S_1,\dots,S_n$ are lin. indep.

$\Rightarrow \sum S_ix\otimes T_iy=0 \quad (x\in H_1,\ y\in H_2)$

Apply $1_{K_1}\otimes\langle\cdot|z\rangle \quad (z\in K_2)$

$\Rightarrow \sum\langle T_iy|z\rangle S_ix=0 \quad (x\in H_1,\ y\in H_2,\ z\in K_2)$

that is, $\sum\langle T_iy|z\rangle S_i=0 \implies$

$\Rightarrow \langle T_iy|z\rangle=0 \quad (i=1,\dots,n,\ y\in H_2,\ z\in K_2)$

$\Rightarrow T_1=\dots=T_n=0 \implies U=0.$ $\square$

Prop. $A,B=*$-algebras, $\pi:A\to\mathcal{B}(H_1)$, $\tau:B\to\mathcal{B}(H_2)$ $*$-reps. Then $\exists$ a $*$-rep $\pi\dot{\otimes}\tau: A\otimes B\to\mathcal{B}(H_1\dot{\otimes}H_2)$ uniquely determined by $(\pi\dot{\otimes}\tau)(a\otimes b)=\pi(a)\dot{\otimes}\tau(b)$ $(a\in A,\ b\in B)$.

Moreover: $\pi\dot{\otimes}\tau$ is faithful $\iff$ π and τ are faithful.

Proof.
$$\begin{tikzcd} A \otimes B \arrow[r, "\pi \otimes \tau"] \arrow[rr, bend right, "\pi \dot{\otimes} \tau"'] & \mathcal{B}(H_1) \otimes \mathcal{B}(H_2) \arrow[r, hook] & \mathcal{B}(H_1 \dot{\otimes} H_2) \end{tikzcd}$$
$\square$

<u>Def</u> A, B = C^*-algebras. The <u>spatial C^*-norm</u> on $A \otimes B$ is given by $\|u\|_* = \|(\omega_A \dot{\otimes} \omega_B)(u)\|$ (where ω_A, ω_B are the univ. reps of A, B).

(<u>пространственная</u> C^*-норма)

Prop $\Rightarrow$ $\|\cdot\|_*$ is indeed a C^*-norm.

The completion of $A \otimes B$ wrt $\|\cdot\|_*$ is denoted by $A \otimes_* B$ and is called the <u>spatial C^*-tens. product</u>.

<u>Observe</u>: $\|a \otimes b\|_* = \|a\| \|b\|$.

<u>Thm.</u> (<u>Takesaki</u>).
$\forall$ C^*-norm $\|\ \|$ on $A \otimes B$ we have $\|\cdot\|_* \leq \|\cdot\|$.

<u>Notation</u> $\otimes_* = \otimes_{min}$ (minimal C^*-tens. prod.)
$\|\cdot\|_* = \|\cdot\|_{min}$.

1. Looks uncomputable.
2. Looks nonfunctorial

$\leftarrow$ quasi-problems with $\otimes_*$

Recall: $A = C^*$-alg, $\pi : A \to \mathcal{B}(H)$ cyclic $*$-rep, $x \in H$ cyclic; $f(a) = \langle \pi(a)x \mid x \rangle \implies$
$\implies f \geq 0$, $\pi \cong \pi_f$.

Lemma. $\|f\| = \|x\|^2$.
In particular, $f \in S(A) \iff \|x\| = 1$.

Proof: CBS $\implies \|f\| \leq \|x\|^2$.
$(e_\lambda) = $ a.i. in $A \implies \pi(e_\lambda)x \to x \implies f(e_\lambda) \to \|x\|^2$.
$\implies \|f\| = \|x\|^2$ $\square$

Thm $A, B = C^*$-algebras $\implies \forall\, u \in A \otimes B$

$$\|u\|_* = \sup\left\{ \|(\pi \dot{\otimes} \tau)(u)\| : \begin{matrix} \pi = *\text{-rep of } A \\ \tau = *\text{-rep of } B \end{matrix} \right\}.$$

Proof We want to show that

$$\|(\pi \dot{\otimes} \tau)(u)\| \leq \|(\omega_A \dot{\otimes} \omega_B)(u)\|.$$

We may assume that π, τ are nondegenerate (because $\pi = \pi' \dot{\oplus} 0$, $\tau = \tau' \dot{\oplus} 0$, where π', τ' are nondegenerate;

$$\|(\pi \dot{\otimes} \tau)(u)\| = \|(\pi' \dot{\otimes} \tau')(u)\| \;).$$

$\pi \cong \dot{\bigoplus}_{i \in I} \pi_i$, $\tau \cong \dot{\bigoplus}_{j \in J} \tau_j$, π_i, τ_j are cyclic.

Lemma $\implies \pi_i \cong \pi_{f_i}$, $\tau_j \cong \pi_{g_j}$, where

$f_i \in S(A)$, $g_j \in S(B)$

$$\pi \dot{\otimes} \tau \cong \dot{\bigoplus_{i,j}} (\pi_i \dot{\otimes} \tau_j) \quad \text{(exer)}$$

$$\|(\pi \dot{\otimes} \tau)(u)\| = \sup_{i,j} \|(\pi_i \dot{\otimes} \tau_j)(u)\| \leq$$

$$\leq \sup_{\substack{f \in S(A) \\ g \in S(B)}} \|(\pi_f \dot{\otimes} \pi_g)(u)\| = \|(\omega_A \dot{\otimes} \omega_B)(u)\| \quad \square$$

<u>Cor.</u> $\varphi : A_1 \to B_1$, $\psi : A_2 \to B_2$ $*$-homoms between C^*-alg. Then $\exists$ a $*$-homom $\varphi \otimes_* \psi : A_1 \otimes_* A_2 \to B_1 \otimes_* B_2$ uniquely determined by $(\varphi \otimes_* \psi)(a_1 \otimes a_2) = \varphi(a_1) \otimes \psi(a_2)$ $\quad (***)$

<u>Proof</u> $\exists$ a $*$-hom $\varphi \otimes \psi : A_1 \otimes A_2 \to B_1 \otimes B_2$ uniquely determined by $(***)$.

We want to show that $\varphi \otimes \psi$ is bdd wrt $\|\cdot\|_*$. $\quad \forall u \in A_1 \otimes A_2$

$$\|(\varphi \otimes \psi)(u)\|_* =$$

$$= \sup \left\{ \|\underbrace{(\pi \dot{\otimes} \tau)(\varphi \otimes \psi)}_{= \pi\varphi \dot{\otimes} \tau\psi}(u)\| : \begin{array}{l} \pi = *\text{-rep of } B_1 \\ \tau = *\text{-rep of } B_2 \end{array} \right\}$$

$$\leq \sup \left\{ \|(\pi_1 \dot{\otimes} \tau_1)(u)\| : \begin{array}{l} \pi_1 = *\text{-rep of } A_1 \\ \tau_1 = *\text{-rep of } A_2 \end{array} \right\}$$

$$= \|u\|_* . \qquad \square$$