GNS reps and cyclic reps

$A = C^*$-alg, $f \in A^*$, $f \geq 0$.

$\pi_f : A \to \mathcal{B}(H_f)$ GNS rep. assoc. to f.

Recall: $\Lambda_f : A \to H_f$, $a \mapsto a + N_f$.

$\pi_f(a)\Lambda_f(b) = \Lambda_f(ab)$; $\langle \Lambda_f a | \Lambda_f b \rangle = f(b^* a)$

Observation Suppose A is unital; $x_f = \Lambda_f(1)$

$\Rightarrow \pi_f(a) x_f = \Lambda_f(a) \Rightarrow x_f$ is cyclic for π_f.

Also, $f(a) = \langle \Lambda_f a | x_f \rangle$.

Lemma. Let (e_λ) be an a.i. in A. Then

(1) (e_λ^2) is an a.i. in A;

(2) $\Lambda_{f_+}(e_\lambda) \longrightarrow \Lambda_{f_+}(1_+)$ in H_{f_+}.

Recall: $f_+ : A_+ \to \mathbb{C}$, $f_+(a + \mu 1_+) = f(a) + \mu \|f\|$.

$f_+ \geq 0$; $\|f_+\| = f_+(1_+) = \|f\|$.

Proof (1) $\|a - a e_\lambda^2\| \leq \|a - a e_\lambda\| + \|(a - a e_\lambda) e_\lambda\| \leq$

$$\leq 2\|a - a e_\lambda\| \longrightarrow 0.$$

(2) $\|\Lambda_{f_+}(1_+ - e_\lambda)\|^2 = f_+((1_+ - e_\lambda)^2) =$

$$= \underbrace{f_+(1_+)}_{=\|f\|} - 2\underbrace{f(e_\lambda)}_{\to \|f\|} + \underbrace{f(e_\lambda^2)}_{\to \|f\|} \longrightarrow 0. \qquad \square$$

Thm. $\exists$ a unique unitary isom. $u: H_f \xrightarrow{\sim} H_{f_+}$ of A-modules s.t. $u(\Lambda_f(a)) = \Lambda_{f_+}(a)$ $(a \in A)$.

Proof. Define $u_0 : H_f^0 \to H_{f_+}^0$, $u_0(a + N_f) = a + N_{f_+}$

u_0 is an A-mod. morph, u_0 is unitary.

Lemma $\Rightarrow$ $\Lambda_{f_+}(1_+) \in \overline{\operatorname{Im} u_0}$. $\Rightarrow$ $\overline{\operatorname{Im} u_0} = H_{f_+}^0$.

$\Rightarrow$ u_0 uniquely extends to a unitary isom

$u : H_f \xrightarrow{\sim} H_{f_+}$ of A-modules. $\square$

Remark.

$$\begin{tikzcd} A \arrow[r, "\pi_f"] \arrow[d, phantom, "\cap"] & \mathcal{B}(H_f) \arrow[d, "\wr", "Ad(u)"'] \\ A_+ \arrow[r, "\pi_{f_+}"] & \mathcal{B}(H_{f_+}) \end{tikzcd}$$

isometric $*$-isom.

$Ad(u)T = uTu^{-1}$.

Convention. Identify H_f and H_{f_+}, $\pi_f(a)$ with $\pi_{f_+}(a)$, $\Lambda_f(a)$ with $\Lambda_{f_+}(a)$ $(a \in A)$

Thm. Define $x_f = \Lambda_{f_+}(1_+) \in H_f$. Then x_f is cyclic for π_f, $\pi_f(a)x_f = \Lambda_f a$, and x_f is uniquely determined by $f(a) = \langle \Lambda_f a | x_f \rangle$.

Proof. $\pi_f(a)x_f = \pi_{f_+}(a)\Lambda_{f_+}(1_+) = \Lambda_f(a)$.

In particular, x_f is cyclic.

$\langle \Lambda_f a | x_f \rangle = \langle \Lambda_{f_+} a | \Lambda_{f_+} 1_+ \rangle = f(a)$

The uniqueness follows from $\overline{\Lambda_f(A)} = H_f$. $\square$

<u>Thm</u> $\pi : A \to \mathcal{B}(H)$ cyclic $*$-rep; $x \in H$ cyclic.
Define $f \in A^*$ by $f(a) = \langle \pi(a) x | x \rangle$.
Then $\exists$ a unique unitary isom. $u : H \xrightarrow{\sim} H_f$ of A-modules s.t. $u(x) = x_f$.

<u>Proof</u>. Let $\Lambda : A \to H$, $\Lambda(a) = \pi(a) x$ $\Longrightarrow$
$\Longrightarrow \overline{\Lambda(A)} = H$; $\langle \Lambda a | \Lambda b \rangle = \langle \pi(b)^* \pi(a) x | x \rangle =$
$= f(b^* a)$; $\pi(a) \Lambda(b) = \Lambda(ab)$ $\Longrightarrow$
$\Rightarrow (H, \pi, \Lambda)$ is an abstr. GNS rep. assoc to f.
$\Rightarrow \exists$ a unitary isom. $u : H \xrightarrow{\sim} H_f$ of A-modules s.t. $u \circ \Lambda = \Lambda_f$. We have $u(\pi(a) x) = \pi_f(a) u(x)$.
Let (e_λ) be an a.i. in A. $\pi_f(e_\lambda) y \to y$ $\forall y$.
$\Longrightarrow \Lambda_{f_+}(e_\lambda) = \pi_f(e_\lambda) u(x) \Longrightarrow x_f = u(x)$.
Uniqueness is clear. $\square$

<u>Exer</u>. $\{\text{Positive } f \in A^*\} \rightleftarrows \left\{ \begin{array}{l} \text{Unitary isom. classes} \\ \text{of } (H, \pi, x) : \pi = \text{cyclic} \\ *\text{-rep of } A \text{ on } H; \\ x \in H \text{ cyclic for } \pi \end{array} \right\}$

Tensor products of C^*-algebras

$A \otimes B \qquad \otimes = \otimes_{\mathbb{C}}$.

1. Multiplication.
2. Involution.
3. C^*-norm. (Nontrivial!)
4. Completion.

<u>Prop.</u> $A, B =$ algebras. $\Rightarrow A \otimes B$ is an alg. w.r.t. the mult. uniquely determined by
$(a_1 \otimes b_1)(a_2 \otimes b_2) = a_1 a_2 \otimes b_1 b_2$.

<u>Proof.</u>
$$\begin{tikzcd} A \times A \arrow[r] \arrow[d] & A \\ A \otimes A \arrow[ur, "\mu_A"'] & \end{tikzcd}$$

$$(A \otimes B) \otimes (A \otimes B) \cong A \otimes A \otimes B \otimes B \xrightarrow[\mu_A \otimes \mu_B]{} A \otimes B$$

Associativity : exer. $\square$

<u>Def.</u> $E =$ vector space over $\mathbb{C}$.
The <u>conjugate space</u> of E is E^c defined as follows: $(E^c, +) = (E, +)$;
λx in $E^c = \bar{\lambda} x$ in E.

<u>Notation:</u> $\forall x \in E$ let x^c be x considered as an elem of E^c.

Then $\lambda x^c = (\bar{\lambda} x)^c$ $(\lambda \in \mathbb{C}, x \in E)$.
That is, $E \to E^c$, $x \mapsto x^c$, is antilinear.

<u>Prop</u> (1) $\mathrm{Hom}_{\mathbb{C}}(E^c, F) = \{\text{antilin. } E \to F\} =$
$= \mathrm{Hom}_{\mathbb{C}}(E, F^c)$

(2) $\{\text{bilin. } E_1^c \times E_2^c \to F\} = \{\text{bilin. } E_1 \times E_2 \to F^c\}$
$= \{\text{antibilin. } E_1 \times E_2 \to F\}$.

(3) $\exists$ a vec. space isom $E_1^c \otimes E_2^c \xrightarrow{\sim} (E_1 \otimes E_2)^c$,
$x_1^c \otimes x_2^c \mapsto (x_1 \otimes x_2)^c$.

<u>Proof</u> (3) It suffices to constr. a natural isom.

$$\mathrm{Hom}_{\mathbb{C}}(E_1^c \otimes E_2^c, F) \cong \mathrm{Hom}_{\mathbb{C}}((E_1 \otimes E_2)^c, F) \quad (F \in \mathrm{Vect})$$

$$\mathrm{Hom}_{\mathbb{C}}(E_1^c \otimes E_2^c, F) \cong \{\text{bilin. } E_1^c \times E_2^c \to F\} =$$
$$= \{\text{bilin. } E_1 \times E_2 \to F^c\} \cong \mathrm{Hom}_{\mathbb{C}}(E_1 \otimes E_2, F^c)$$
$$\cong \mathrm{Hom}_{\mathbb{C}}((E_1 \otimes E_2)^c, F) \qquad \square$$

Prop. $A, B = *$-algebras $\Rightarrow A \otimes B$ is a $*$-alg. w.r.t the invol. uniquely determined by $(a \otimes b)^* = a^* \otimes b^*$ $(a \in A, b \in B)$.

Proof. $i_A : A \to A^c$, $a \mapsto a^*$, is linear.

$$A \otimes B \xrightarrow[i_A \otimes i_B]{} A^c \otimes B^c \cong (A \otimes B)^c.$$

Exer: this is an invol. on $A \otimes B$. $\square$.

Def. $A = *$-alg. A norm $\|\cdot\|$ on A is a <u>C^*-norm</u> if

(1) $\|ab\| \le \|a\| \|b\|$;

(2) $\|a^*\| = \|a\|$; $\qquad (a, b \in A)$

(3) $\|a^* a\| = \|a\|^2$

$(A, \|\cdot\|)$ is a <u>pre-C^*-algebra</u>.

<u>Observe</u>: the completion of a pre-C^*-alg is a C^*-alg.

Prop. $A = *$-alg; $\|\cdot\|$ and $\|\cdot\|_1$ $= C^*$-norms on A. Suppose $(A, \|\cdot\|)$ is complete. Then $\|\cdot\| = \|\cdot\|_1$.

Proof. Let A_1 be the completion of $(A, \|\cdot\|_1)$.

$j : \underset{\|\cdot\|}{A} \hookrightarrow \underset{\|\cdot\|_1}{A_1}$ is an inj. $*$-hom btw C^*-alg

$\Rightarrow j$ is isometric $\square$.

<u>Notation</u> A = algebra

$M_n(A) = \{ n\times n\text{-matrices with entries in } A\}$.

$M_n = M_n(\mathbb{C})$

$$M_n \otimes A \xrightarrow[\text{alg. isomorphism}]{\sim} M_n(A)\ ; \quad (\alpha_{ij})\otimes a \mapsto (\alpha_{ij}a).$$

<u>Prop.</u> $A = C^*$-algebra. $\Rightarrow \exists$ a unique C^*-norm on $M_n\otimes A$. This norm makes $M_n\otimes A$ into a C^*-alg.

<u>Proof</u> Let $\pi : A \to \mathcal{B}(H)$ be a faithful $*$-rep.

$$M_n\otimes A \xrightarrow[\text{inj. }*\text{-hom}]{1\otimes\pi} M_n\otimes\mathcal{B}(H) \cong M_n(\mathcal{B}(H)) = \mathcal{B}(H^n)$$

C^*-alg.

Let $\|u\| = \|(1\otimes\pi)(u)\|$. $(u\in M_n\otimes A)$

This is a C^*-norm.

$\mathrm{Im}(1\otimes\pi) = \{u\in\mathcal{B}(H^n) : u_{ij}\in \mathrm{Im}\,\pi\ \forall i,j\}$

is closed in $\mathcal{B}(H^n)$ (exer.)

$\Rightarrow (M_n\otimes A, \|\cdot\|)$ is a C^*-alg.

Uniqueness follows from the previous prop. $\square$