GNS construction
(a summary)

$A = C^*\text{-alg}, \ f \in A^*, \ f \geq 0.$

$\langle a | b \rangle = f(b^* a)$ pre-inner prod. on A

$\|a\|_f = f(a^* a)^{1/2}$

$$N_f = \{a \in A : \|a\|_f = 0\} = \{a \in A : f(ba) = 0 \ \forall b \in A\}$$
$$= \{a \in A : f(a^* b) = 0 \ \forall b \in A\}.$$

$\langle a + N_f | b + N_f \rangle = \langle a | b \rangle = f(b^* a)$ inn. prod. on A/N_f.

$H_f^0 = (A/N_f, \langle \cdot | \cdot \rangle)$

$$\begin{tikzcd} H_f^0 \arrow[r, "\pi_f^0(a)"] \arrow[d, phantom, "\cap"] & H_f^0 \arrow[d, phantom, "\cap"] \\ H_f \arrow[r, "\pi_f(a)"] & H_f \end{tikzcd}$$

$b + N_f \mapsto ab + N_f.$

$\pi_f : A \to \mathcal{B}(H_f)$

GNS representation

$\Lambda_f : A \to H_f, \quad a \mapsto a + N_f.$

- $\overline{\Lambda_f(A)} = H_f$;
- $\langle \Lambda_f a | \Lambda_f b \rangle = f(b^* a)$
- $\pi_f(a) \Lambda_f(b) = \Lambda_f(ab)$

(H, π, Λ)

$$\begin{tikzcd} H \arrow[rr, "\sim"] & & H_f \\ & A \arrow[ul, "\Lambda"] \arrow[ur, "\Lambda_f"'] & \end{tikzcd}$$

The universal representation

The 2nd Gelfand-Naimark Thm.

1 Summable families

X = normed sp; I = a set; $(x_i)_{i\in I}$ a family of elements of X.

$$\sum_{i\in I} x_i = ?$$

$\mathrm{Fin}(I) = \{J \subset I : J \text{ is finite}\}$.

$(\mathrm{Fin}(I), \subseteq)$ is a dir. poset.

$\forall\, J \in \mathrm{Fin}(I)$ $\quad x_J = \sum_{i\in J} x_i$

Def. (x_i) is <u>summable</u> $\iff$ the net $(x_J)_{J\in \mathrm{Fin}(I)}$ converges in X. $\quad \sum_{i\in I} x_i = \lim_{\mathrm{Fin}(I)} x_J$.

Exer. $\sum_{i\in I} x_i = x \in X \iff$

(1) $I_0 = \{i \in I : x_i \neq 0\}$ is at most countable;

(2) If I_0 is finite, then $\sum_{i\in I_0} x_i = x$, and if I_0 is infinite, then $\forall$ bij. $\varphi : \mathbb{N} \to I_0$ the series $\sum_n x_{\varphi(n)}$ converges to x.

2. Hilbert direct sums

$\{H_i : i\in I\}$ a family of Hilbert spaces.

$$H = \Big\{ x=(x_i)\in \prod_{i\in I} H_i : \sum_{i\in I} \|x_i\|^2 < \infty \Big\}.$$

Exer. (1) H is a vec. subspace of $\prod_{i\in I} H_i$;

(2) H is a Hilb. space w.r.t. $\langle x|y\rangle = \sum_{i\in I} \langle x_i|y_i\rangle$.

Def H is the Hilbert direct sum of $(H_i)_{i\in I}$;

$$H = \dot{\bigoplus_{i\in I}} H_i .$$

Prop. A = Banach $*$-alg, $\{H_i : i\in I\}$ a family of $*$-modules over A. Then $\dot{\bigoplus}_{i\in I} H_i$ is a $*$-module over A w.r.t.

$$a\cdot (x_i)_{i\in I} = (ax_i)_{i\in I}.$$

Proof. $\|ax_i\| \le \|a\|\|x_i\| \ \forall i \Rightarrow ax \in \dot{\bigoplus}_{i\in I} H_i$

$(\forall a\in A,\ x\in \dot{\bigoplus}_{i\in I} H_i)$. $\square$

Notation. $\{\pi_i : A \to \mathcal{B}(H_i)\}_{i\in I}$ a family of $*$-reps of A. The $*$-rep of A assoc. to $\dot{\bigoplus}_{i\in I} H_i$ is called the Hilb. dir. sum of $\{\pi_i\}_{i\in I}$ and is denoted by $\dot{\bigoplus}_{i\in I} \pi_i$.

3. The universal representation

$A = C^*$-alg.

<u>Def</u> The <u>universal representation</u> of A is $\omega_A = \dot{\bigoplus}_{f \in S(A)} \pi_f$.

<u>Thm</u> (<u>2nd Gelfand-Naimark thm</u>)

Every C^*-alg has a faithful (= isometric) $*$-representation. Specifically, ω_A is faithful.

<u>Lemma</u> $\forall a \in A \ \exists f \in S(A)$ s.t. $\|\pi_f(a)\| = \|a\|$.

<u>Proof</u>. Assume $a \neq 0$.

$\exists f \in S(A)$ s.t. $f((aa^*)^2) = \|(aa^*)^2\|$.

$$\|a\|^4 = \|a a^*\|^2 = \|(aa^*)^2\| = f((aa^*)^2) =$$
$$= \|\Lambda_f(aa^*)\|^2 = \|\pi_f(a)\Lambda_f(a^*)\|^2 \leq$$
$$\leq \|\pi_f(a)\|^2 f(aa^*) \leq \|\pi_f(a)\|^2 \|a\|^2 \implies$$
$$\implies \|\pi_f(a)\| \geq \|a\|. \qquad \square$$

<u>Proof of Thm</u>.

$$\|a\| \overset{(L.)}{\leq} \sup_{f \in S(A)} \|\pi_f(a)\| = \|\omega_A(a)\| \leq \|a\|. \qquad \square$$

Exer. A = separable C^*-alg. Then
(1) $\forall f \in A^*$, $f \geq 0$, H_f is separable.
(2) $\exists f \in S(A)$ s.t. π_f is faithful.

Nondegenerate reps, cyclic reps, GNS reps.

$A = C^*$-alg; $H = *$-module over A.

$AH = \operatorname{span}\{ax \mid a \in A, x \in H\} \subset H$ submodule.

Def. H is essential (nondegenerate) $\Longleftrightarrow$
$\Longleftrightarrow \overline{AH} = H$.

$H_{ess} = \overline{AH}$ is the essential part of H.

Prop. H_{ess} is a closed essential submodule of H, and the action of A on $H_{ess}^{\perp}$ is trivial (i.e., $ax = 0$ $\forall a \in A$, $\forall x \in H_{ess}^{\perp}$).

Proof. A has an a.i. $\Rightarrow A = \overline{A^2} = \overline{\operatorname{span}\{ab : a, b \in A\}}$.

$\Rightarrow \overline{AH_{ess}} = \overline{A\overline{AH}} = \overline{AAH} = \overline{\overline{AA}H} = \overline{AH} = H_{ess}$.

$\forall a \in A$, $\forall x \in H_{ess}^{\perp}$

$\langle ax \mid ax \rangle = \langle x \mid \underbrace{a^*ax}_{\in H_{ess}} \rangle = 0 \Rightarrow ax = 0$. $\square$

Def. The <u>cyclic submodule of H generated by $x \in H$</u> is $\overline{Ax} = \overline{\{ax : a \in A\}}$.

This is a closed submodule of H

Prop. $A = C^*$-alg, (e_λ) an a.i. in A, $H = *$-mod over A. Then $H_{ess} = \{x \in H : x = \lim\limits_\lambda e_\lambda x\}$.

Lemma. E, F = Banach spaces; (T_λ) = a bdd net in $\mathcal{B}(E,F)$; $M \subset E$ total subset (i.e., $\overline{\mathrm{span}(M)} = E$). Suppose $\forall x \in M$ $\exists \lim\limits_\lambda T_\lambda x \in F$. Then $\exists T \in \mathcal{B}(E,F)$ s.t. $\forall x \in E$ $T_\lambda x \to Tx$.

Proof: exer.

Proof of Prop. $(\supset)$ clear

$(\subset)$: $x = \lim e_\lambda x$ holds $\forall x = ay$ $(a \in A, y \in H)$

By Lemma, the same holds $\forall x \in H_{ess}$. $\square$

Cor. If $x \in H_{ess}$, then $x \in \overline{Ax}$.

Proof. $x = \lim\limits_\lambda e_\lambda x \in \overline{Ax}$. $\square$

Exer. H = $*$-mod over A (A = Ban. $*$-alg); $\{H_i\}$ = a family of closed pairwise orth. submodules of H. Then $\exists$ a unitary isom of A-modules

$$\dot{\bigoplus_{i\in I}} H_i \longrightarrow \overline{\bigoplus_{i\in I} H_i} \subset H, \quad (x_i)_{i\in I} \mapsto \sum_{i\in I} x_i$$

If $\overline{\bigoplus H_i} = H$, then we say that H is the <u>Hilb. dir. sum</u> of $\{H_i\}$.

Thm. A = C^*-alg, H = a nondegenerate $*$-module over A. Then $\exists$ a family $\{H_i : i\in I\}$ of closed pairwise orth cyclic submodules of H s.t. $H = \dot{\bigoplus_{i\in I}} H_i$

Proof $M = \left\{ \begin{array}{c} \text{families of closed, pairwise orth} \\ \text{nonzero cyclic submod of } H \end{array} \right\}$

$(M, \subset)$ satisfies the conditions of Zorn's lemma.

Let $\{H_i : i\in I\}$ be a max. element of M;

$H_0 = \overline{\bigoplus_{i\in I} H_i}$. $\qquad H = H_0 \oplus H_0^\perp$.

Suppose $H_0^\perp \neq 0$. Take any $x \in H_0^\perp$, $x \neq 0$.

$\{H_i : i \in I\} \cup \{\overline{Ax}\} \in M$, a contradiction.

$\begin{array}{c} \rotatebox{90}{$\in$} \\ x \end{array}$ $\Longrightarrow H_0^{\perp} = 0$. $\square$