Positive functionals

$A = *$-alg, $f : A \to \mathbb{C}$ lin.

Def f is positive ($f \geq 0$) $\iff f(a^*a) \geq 0 \ \forall a \in A$.

We know:

(1) $A = C^*$-alg. $f \geq 0 \iff f(A_{pos}) \subset [0, +\infty)$

(2) $A = C^*$-alg, $f \geq 0 \implies f(a^*) = \overline{f(a)}$ $(a \in A)$;
f is cont.

$A = *$-alg, $f : A \to \mathbb{C}$, $f \geq 0$

Notation $\langle a | b \rangle_f = f(b^*a)$ $(a, b \in A)$

$\langle \cdot | \cdot \rangle_f$ is a sesquilin. form;

$\langle a | a \rangle_f = f(a^*a) \in \mathbb{R} \underset{\text{(exer)}}{\implies} \langle \cdot | \cdot \rangle_f$ is Hermitian,

that is, $\langle b | a \rangle_f = \overline{\langle a | b \rangle_f}$.

Hence $\langle \cdot | \cdot \rangle_f$ is a pre-inner product on A.

Prop (CBS ineq.)

$|f(b^*a)|^2 \leq f(a^*a) f(b^*b)$ Equivalently,

$|f(ab)|^2 \leq f(aa^*) f(b^*b)$ $(a, b \in A)$

Notation $A = C^*\text{-alg}$, $f \in A^*$.

$f_+ : A_+ \to \mathbb{C}$, $f_+(a + \lambda 1_+) = f(a) + \lambda \|f\|$. $f_+ \in A_+^*$.

Thm. $A = C^*\text{-alg}$; (e_λ) an a.i. in A; $f \in A^*$.

TFAE:

(1) $f \geq 0$;

(2) $\lim f(e_\lambda) = \|f\|$;

(3) $f_+ \geq 0$.

Lemma $A =$ unital $C^*\text{-alg}$, $f \in A^*$, $f(1) = \|f\|$
$\Rightarrow f \geq 0$.

Proof. We may assume that $\|f\| = f(1) = 1$.

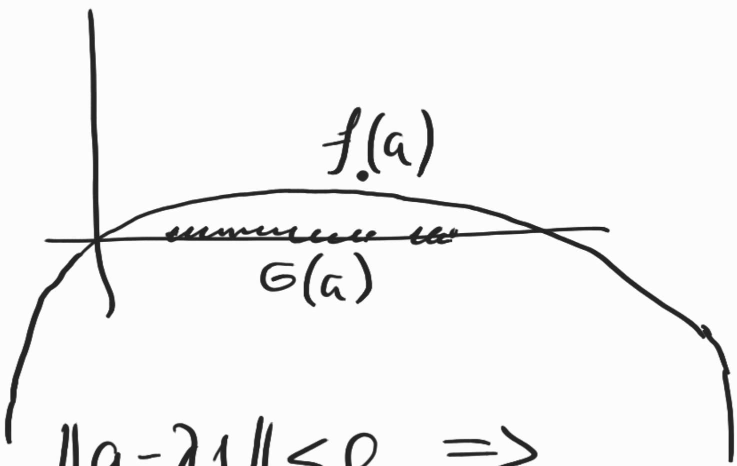
If $f \ngeq 0$, then $\exists\, a \in A_{pos}$ s.t. $f(a) \notin [0, +\infty)$.

$\sigma(a) \subset [0, +\infty)$ is compact.

$\exists$ a closed disc $D \subset \mathbb{C}$ s.t. $\sigma(a) \subset D$, $f(a) \notin D$.

$D = D(\lambda, \rho) =$

$= \{ z \in \mathbb{C} : |z - \lambda| \leq \rho \}$.



$r(\underbrace{a - \lambda 1}_{\text{normal}}) = \|a - \lambda 1\|$;

$\sigma(a - \lambda 1) \subset D(0, \rho) \Longrightarrow \|a - \lambda 1\| \leq \rho \Longrightarrow$

$\Rightarrow \underset{\displaystyle |f(a) - \lambda|}{\underset{\parallel}{|f(a - \lambda 1)|}} \leq \rho \Rightarrow f(a) \in D$ a contr. $\square$

Proof of Thm. We may assume that $\|f\|=1$

$(1) \Rightarrow (2)$. $e_\lambda \nearrow \Rightarrow f(e_\lambda) \nearrow$; $\|e_\lambda\| \le 1 \Rightarrow$

$\Rightarrow \exists \lim f(e_\lambda) \le 1$.

Observe: $\forall a \in A$, $\|a\| \le 1$, we have $0 \le f(a^*a) \le 1$

$$|f(e_\lambda a)|^2 \underset{(CBS)}{\le} f(e_\lambda^2) f(a^*a) \le f(e_\lambda)$$

(because $e_\lambda^2 \le e_\lambda$ by 1GN)

$\Rightarrow |f(a)|^2 \le \lim f(e_\lambda) \Rightarrow \|f\| = 1 \le \lim f(e_\lambda)$

$\Rightarrow \lim f(e_\lambda) = 1$.

$(2) \Rightarrow (3)$ $f_+(1_+) = 1 \Rightarrow$ it suff. to show that $\|f_+\| = 1$. (hence $f_+ \ge 0$ by Lemma)

Clearly, $\|f_+\| \ge 1$.

$\forall a \in A$, $\forall \mu \in \mathbb{C}$

$$|f_+(a + \mu 1_+)| = |f(a) + \mu| = |\lim_\lambda f(a e_\lambda) + \mu \lim_\lambda f(e_\lambda)|$$

$$= |\lim_\lambda f((a + \mu 1_+) e_\lambda)| \le \sup_\lambda |f((a + \mu 1_+) e_\lambda)|$$

$$\le \|a + \mu 1_+\|.$$

$\Rightarrow \|f_+\| \le 1 \Rightarrow \|f_+\| = 1 \underset{L.}{\Longrightarrow} f_+ \ge 0$.

$(3) \Rightarrow (1)$ clear. $\square$

Cor. A = unital C^*-alg, $f \in A^*$

$f \geq 0 \iff \|f\| = f(1)$.

Thm. $A = C^*$-alg, $B \subset A$ closed $*$-subalg;

$g \in B^*$, $g \geq 0 \Rightarrow \exists f \in A^*$ s.t. $f|_B = g$, $\|f\| = \|g\|$.

$f \geq 0$

Proof We may assume that A, B are unital, and $1_A \in B$.

(otherwise consider A_+, B_+, g_+)

Hahn-Banach $\Rightarrow \exists f \in A^*$ s.t. $f|_B = g$, $\|f\| = \|g\|$

$\Rightarrow \|f\| = \|g\| = g(1) = f(1) \Rightarrow f \geq 0$. $\square$

Cor. $A = C^*$-alg, $a \in A$ normal $\Rightarrow$

$\Rightarrow \exists f \in S(A)$ s.t. $|f(a)| = \|a\|$.

Proof Case 1: $A = C_0(X)$, X = loc. comp. Hausd. space

$\Rightarrow \exists x \in X$ s.t. $|a(x)| = \|a\|$. Let $f = \varepsilon_x$.

General case: consider $B = C^*_A(a)$ and apply Thm. $\square$

$*$-representations and $*$-modules

$A = *$-alg, H = Hilb. space.

<u>Def</u> A <u>$*$-representation</u> of A on H is a $*$-hom $\pi : A \longrightarrow \mathcal{B}(H)$

π is <u>faithful</u> $\iff \operatorname{Ker}\pi = 0$.

<u>Recall</u>: (1) A = Banach $*$-alg $\Rightarrow \pi$ is cont, $\|\pi\| \le 1$.

(2) $A = C^*$-alg, π is faithful $\Rightarrow \pi$ is isometr.

<u>Def.</u> A <u>left $*$-module</u> over A is a left A-mod H together with an inner product which makes H into a Hilb. space and satisfies

$\langle ax | y \rangle = \langle x | a^* y \rangle \quad (x, y \in H,\ a \in A)$

<u>Warning</u> $*$-modules $\neq$ Hilbert C^*-modules

<u>Exer.</u> $\{ *\text{-reps of } A \} \underset{\longleftarrow}{\overset{1-1}{\longrightarrow}} \left\{ \begin{matrix} *\text{-modules} \\ \text{over } A \end{matrix} \right\}$

$ax = \pi(a)x \quad (a \in A,\ x \in H)$

(Important: $\pi(a)$ is bdd!)

<u>Def</u> H_1, H_2 $*$-modules over A.

A <u>morphism</u> $\varphi : H_1 \to H_2$ is a cont. A-mod homom.

(i.e., $\varphi(ax) = a\varphi(x)\ \forall x \in H_1,\ a \in A$)

Terminology morphisms = intertwining maps
(сплетающий оп-р)

Prop. A = $*$-alg, H = left $*$-mod over A, $H_0 \subset H$ submodule $\Rightarrow$ $H_0^\perp$ is a submod of H.

Proof. $\forall a \in A$, $\forall x \in H_0^\perp$, $\forall y \in H_0$

$$\langle ax | y \rangle = \langle x | \underbrace{a^* y}_{\in H_0} \rangle = 0 \Rightarrow ax \in H_0^\perp \quad \square$$

GNS construction
(Gelfand, Naimark, Segal)

A = $*$-alg, $f : A \to \mathbb{C}$, $f \geq 0$.

Recall: $\langle a | b \rangle_f = f(b^* a)$ is a pre-inner product on A

$\Rightarrow$ $\|a\|_f = \sqrt{\langle a | a \rangle_f}$ is a seminorm on A.

Notation $N_f = \{ a \in A : \|a\|_f = 0 \} = \{ a \in A : f(a^* a) = 0 \}$

Lemma. $N_f = \{ a \in A : f(ba) = 0 \ \forall b \in A \}$
$= \{ a \in A : f(a^* b) = 0 \ \forall b \in A \}$.

Proof. $|f(ba)|^2 \leq f(bb^*) f(a^* a)$

$|f(a^* b)|^2 \leq f(a^* a) f(b^* b) \quad \square$

Cor. 1 $\langle a+N_f | b+N_f \rangle = \langle a|b\rangle_f = f(b^*a)$
is a well-defined inner product on A/N_f.

Notation $H_f^o = (A/N_f, \langle \cdot | \cdot \rangle)$.
$H_f =$ completion of H_f^o. H_f is a Hilb. space.

Cor. 2. N_f is a left ideal of A.

This implies that $A/N_f = H_f^o$ is a left A-mod:
$$a(b+N_f) = ab + N_f.$$
Let $\pi_f^o : A \to \mathrm{End}_{\mathbb{C}}(H_f^o)$ denote the respective representation; $\pi_f^o(a)(b+N_f) = ab + N_f$.

Prop $A = C^*$-alg, $f \in A$, $f \geq 0$. Then
$\forall a \in A$ $\pi_f^o(a)$ is bdd, $\|\pi_f^o(a)\| \leq \|a\|$.

Proof
$$\|\pi_f^o(a)(b+N_f)\|^2 = \|ab+N_f\|^2 = f(b^*a^*ab).$$
Observe: $0 \leq a^*a \leq \|a^*a\| 1_+$. $(1 \in N)$.
$\Rightarrow b^*a^*ab \leq \|a\|^2 b^*b \Rightarrow$
$\Rightarrow f(b^*a^*ab) \leq \|a\|^2 f(b^*b) = \|a\|^2 \|b+N_f\|^2$. $\square$

<u>Thm</u> $A = C^*$-alg, $f \in A^*$, $f \geq 0$. Then

(1) $\forall a \in A$ $\pi_f^0(a)$ uniquely extends to $\pi_f(a) \in \mathcal{B}(H_f)$, and $\|\pi_f(a)\| \leq \|a\|$

(2) $\pi_f : A \to \mathcal{B}(H_f)$ is a $*$-rep of A.

<u>Proof</u> (1) follows from Prop.

(2) Clearly, $\pi_f(ab) = \pi_f(a)\pi_f(b)$ $\forall a, b \in A$

$$\langle \pi_f(a)(b+N_f) \mid c+N_f \rangle = \langle ab+N_f \mid c+N_f \rangle =$$

$$= f(c^*ab) = f((a^*c)^*b) = \langle b+N_f \mid a^*c+N_f \rangle$$

$$= \langle b+N_f \mid \pi_f(a^*)(c+N_f) \rangle \Rightarrow \pi_f(a)^* = \pi_f(a^*) \quad \square$$

<u>Def</u> π_f is the <u>GNS representation</u> associated to f.

<u>Notation.</u> $\Lambda_f : A \to H_f$, $a \in A \mapsto a + N_f$

<u>Properties of Λ_f</u>:

(1) $\overline{\Lambda_f(A)} = H_f$;

(2) $\langle \Lambda_f a \mid \Lambda_f b \rangle = f(b^*a)$ $(a, b \in A)$

(3) $\pi_f(a)\Lambda_f(b) = \Lambda_f(ab)$

Def. An <u>abstract GNS representation</u> of A associated to f is (H, π, Λ) where H = Hilb. space, $\pi : A \to \mathcal{B}(H)$ is a $*$-rep, $\Lambda : A \to H$ linear s.t. (1)–(3) hold.

Prop. $A = C^*$-alg, $f \in A^*$, $f \geq 0$;
(H, π, Λ) an abstract GNS rep assoc. to f.
Then $\exists$ a unique unitary isom. $u : H \to H_f$ of A-modules s.t. $u \circ \Lambda = \Lambda_f$.

Proof: exer. $\square$

Exer. Describe (H_f, π_f, Λ_f) explicitly in the foll. cases:

(1) $A = C_0(X)$, μ = pos. finite Radon meas on X,
$f(a) = \int_X a \, d\mu$.

(2) $A = M_n(\mathbb{C})$, $f(T) = \frac{1}{n} Tr(T)$

(3) $A = \mathcal{B}(H)$ or $A = \mathcal{K}(H)$, $\xi \in H$, $\|\xi\| = 1$
$f(T) = \langle T\xi | \xi \rangle$.