Approximate identities in C^*-algebras

A = normed alg.

Def A net $(e_\lambda)_{\lambda\in\Lambda}$ in A is an approximate identity (a.i.) $\iff$ $\forall a \in A$ $\;ae_\lambda \to a$, $e_\lambda a \to a$.

Thm Every C^*-alg A has an a.i. $(e_\lambda)_{\lambda\in\Lambda}$ s.t.

(AI1) $0 \le e_\lambda \le 1_+$ $\forall \lambda$

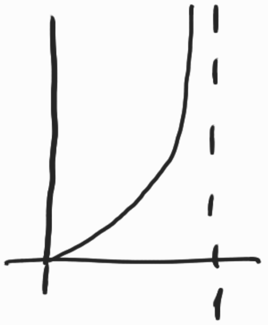
(equivalently, $e_\lambda \ge 0$ and $\|e_\lambda\| \le 1$)

(AI2) $\lambda \le \mu \Rightarrow e_\lambda \le e_\mu$.

Proof $\Lambda = \{e \in A_{pos} : \|e\| < 1\}$ $(\Lambda, \le)$ is a poset

Claim 1. $\Lambda \cong A_{pos}$ as a poset. As a corollary, Λ is directed.

Proof Consider $[0,1) \underset{\psi}{\overset{\varphi}{\rightleftarrows}} [0,+\infty)$



$\varphi(t) = \dfrac{1}{1-t} - 1$ $\qquad$ $\psi(s) = \varphi^{-1}(s) = 1 - \dfrac{1}{1+s}$.

$\forall\, e \in \Lambda \quad \varphi(e) \in A_{pos}$

$e_1, e_2 \in \Lambda,\ e_1 \le e_2 \Rightarrow 1_+ - e_1 \ge 1_+ - e_2$

$\Rightarrow (1_+ - e_2)^{-1} \le (1_+ - e_1)^{-1} \Rightarrow \varphi(e_1) \le \varphi(e_2)$

$$\boxed{\begin{array}{l} 0 \le a \le b \\ \Rightarrow b^{-1} \le a^{-1} \end{array}}$$

$\forall\, a \in A_{pos} \quad \psi(a) \in \Lambda$ (Spec. Map. Property)

$a_1 \le a_2 \Rightarrow \psi(a_1) \le \psi(a_2)$ (exer.)

$\Lambda \underset{\psi}{\overset{\varphi}{\rightleftarrows}} A_{pos}$ Superpos. property $\Rightarrow$

$\varphi\psi = 1_{A_{pos}},\ \psi\varphi = 1_\Lambda$. $\square$

$\Lambda \hookrightarrow A$, $e \mapsto e$, is a net satisfying (AI1), (AI2).

<u>Claim 2.</u> $\forall\, a \in A_{pos}$, $\|a\| \le 1$, we have $\|a(1_+ - e)a\| \to 0$ $(e \in \Lambda)$

<u>Proof.</u> Consider $t : [0,1] \hookrightarrow \mathbb{C}$, $t(\lambda) = \lambda$.

$t(1 - t^{1/n}) \to 0$ in $C[0,1]$ (exer.)

$\Rightarrow a(1_+ - a^{\frac{1}{n}}) \to 0$ in A.

$\forall\, \varepsilon > 0\ \exists\, n \in \mathbb{N}$ s.t. $\|a(1_+ - a^{1/n})\| < \varepsilon$.

$\forall\, e \in \Lambda$ s.t. $e \ge a^{1/n}$ $\|a(1_+ - e)a\| \le \|a(1_+ - a^{\frac{1}{n}})a\| < \varepsilon$

$\square$

$$\boxed{a \le b \Rightarrow x^* a x \le x^* b x}$$

Claim 3. $(e)_{e\in\Lambda}$ is an a.i. in A.

Proof. $\forall\, e\in\Lambda \quad (1_+-e)^2\le 1_+-e \quad$ (1GN Thm)

$\Rightarrow \forall\, a\in A_{pos},\ \|a\|\le 1$

$\|a-ea\|^2=\|(1_+-e)a\|^2=\|a(1_+-e)^2a\|\le$

$\le\|a(1_+-e)a\|\to 0$

$\Rightarrow a-ea\to 0\ (e\in\Lambda)$ for all $a\in A$ (by linearity);

$a-ae=(a^*-ea^*)^*\to 0.$ $\square\square$

Exer. A = separable C^*-alg $\Rightarrow$ A has a sequential a.i.

Convention. "a.i." will mean "a.i. satisfying (AI1) and (AI2)"

Quotient C^*-algebras

Observations:

(1) A = Ban. alg, $I\subset A$ closed 2-sided ideal
$\Rightarrow A/I$ is a Banach alg;
$\|a+I\|=\inf\{\|a+b\|: b\in I\}$.

(2) A = Ban. $*$-alg, $I\subset A$ closed 2-sided $*$-ideal
$\Rightarrow A/I$ is a Banach $*$-alg; $(a+I)^*=a^*+I$.

Lemma. $A = C^*$-alg, $I \subset A$ closed left ideal
$\Rightarrow$ I has a right a.i. (e_λ) satisfying (AI1), (AI2)
(i.e., a net (e_λ) s.t. $ae_\lambda \to a$ $(\lambda \in \Lambda)$)

Proof. Let $B = I \cap I^* \Rightarrow B$ is a C^*-subalg of A.
Let (e_λ) be an a.i. in B.

$\forall\, a \in I \quad \|a - ae_\lambda\|^2 = \|a(1_+ - e_\lambda)\|^2 =$

$= \|(1_+ - e_\lambda)a^*a(1_+ - e_\lambda)\| \le \|\underbrace{a^*a}_{\in B}(1_+ - e_\lambda)\| \to 0.$ $\square$

Thm. $A = C^*$-alg, $I \subset A$ closed 2-sided ideal.
Then

(1) I is a $*$-ideal

(2) Suppose (e_λ) is an a.i. in I. Then $\forall\, a \in A$

$$\|a + I\| = \lim_\lambda \|a - ae_\lambda\| = \inf_\lambda \|a - ae_\lambda\|.$$

(3) A/I is a C^*-algebra.

Proof. (1) Let (e_λ) be a right a.i. in I satisfying (AI1), (AI2)

$\forall\, a \in I \quad ae_\lambda \to a \Rightarrow \underbrace{e_\lambda a^*}_{\in I} \to a^* \Rightarrow a^* \in I.$

(2) $\forall a \in A$, $\forall b \in I$

$$\|a - ae_\lambda\| = \|a(1_+ - e_\lambda)\| \le \|(a+b)(1_+ - e_\lambda)\| + \|b(1_+ - e_\lambda)\| \le \|a+b\| + \underbrace{\|b - be_\lambda\|}_{\to 0}$$

$$\Rightarrow \overline{\lim}\, \|a - ae_\lambda\| \le \|a + I\|$$

$$\|a + I\| \le \inf_\lambda \|a - ae_\lambda\| \le \underline{\lim}\, \|a - ae_\lambda\|. \Rightarrow (2).$$

(3) $\forall a \in A$.

$$\|a + I\|^2 = \inf_\lambda \|a(1_+ - e_\lambda)\|^2$$

$$= \inf_\lambda \|(1_+ - e_\lambda)a^*a(1_+ - e_\lambda)\| \le \inf_\lambda \|a^*a(1_+ - e_\lambda)\|$$

$$= \|a^*a + I\|. \quad \square$$

Example 1 (Calkin algebra)

H = Hilb. space; $Q(H) = \mathcal{B}(H)/\mathcal{K}(H)$ is a C^*-algebra (the Calkin algebra)

Example 2 (Toeplitz algebra)

H = inf-dim. separable Hilb. space;

$\{e_n : n \in \mathbb{Z}_{\ge 0}\}$ an ON basis in H;

$S \in \mathcal{B}(H)$ $\quad Se_n = e_{n+1}$ $\forall n$ (right shift oper.)

Def. The Toeplitz algebra is the C^*-subalg $\mathcal{T}$ of $\mathcal{B}(H)$ generated by S.

Explicitly: $\mathcal{T} = \overline{\operatorname{span}}\{ S^k (S^*)^\ell : k, \ell \in \mathbb{Z}_{\geq 0}\}$.
(because $S^* S = 1$)

Exer. (1) $\mathcal{K}(H) \subset \mathcal{T}$

(2) $\mathcal{T}/\mathcal{K}(H) \cong C(\mathbb{T})$ $\quad$ $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.

(3) The Toeplitz extension

$$0 \to \mathcal{K}(H) \hookrightarrow \mathcal{T} \xrightarrow[p]{} C(\mathbb{T}) \longrightarrow 0$$

does not split (i.e., $\nexists$ $*$-alg. hom. $j : C(\mathbb{T}) \to \mathcal{T}$ s.t. $pj = 1_{C(\mathbb{T})}$).

Positive functionals

A = $*$-alg. $f : A \to \mathbb{C}$ linear.

Def. f is positive ($f \geq 0$) $\iff$ $f(a^* a) \geq 0$ $\forall a \in A$.

If A is Banach $*$-alg, then $f \in A^*$ is a state $\iff$ $f \geq 0$ and $\|f\| = 1$. (состояние)

Observe: If A is a C^*-alg, then
$$f \geq 0 \iff f(A_{pos}) \subset [0, +\infty)$$

Example 1 $\chi : A \to \mathbb{C}$ is a $*$-char $\Rightarrow \chi \geq 0$.

If A is a C^*-alg, $\chi \neq 0 \Rightarrow \chi$ is a state.

Example 2. $A = C_0(X)$, X = loc. comp. Hausd. space.

$\forall$ fin. positive Radon measure μ on X let

$f_\mu : A \to \mathbb{C}$ $\quad f_\mu(a) = \int_X a \, d\mu$. $\quad f_\mu \geq 0$;

$\|f_\mu\| = \mu(X)$ (exer). $\quad f_\mu$ is a state $\Leftrightarrow \mu(X) = 1$.

Riesz Rep. Thm:

$$\left\{ \begin{matrix} \text{Fin. positive} \\ \text{Radon measures on } X \end{matrix} \right\} \underset{\longleftarrow}{\overset{1-1}{\longrightarrow}} \left\{ \begin{matrix} \text{Pos. linear func} \\ \text{on } C_0(X) \end{matrix} \right\}$$

$$\mu \mapsto f_\mu.$$

(exer)

Example 3 $A = M_n(\mathbb{C}) = \mathcal{B}(H)$ $\quad \dim H = n$.

(1) $f : A \to \mathbb{C}$, $f(T) = \frac{1}{n} Tr(T)$ (normalized trace)

is a state.

(2) $\forall$ lin. $f : A \to \mathbb{C}$ $\exists$ a unique $S \in A$ s.t.

$f(T) = Tr(TS)$ $(T \in A)$; $f = f_S$.

(3) $f_S \geq 0 \Leftrightarrow S \geq 0$;

(4) f_S is a state $\Leftrightarrow S \geq 0$ and $Tr(S) = 1$.

Example 4 $A = \mathcal{B}(H)$ $\quad H = $ Hilb. space

$\xi \in H \quad f_\xi : A \to \mathbb{C} \quad f_\xi(T) = \langle T\xi | \xi \rangle.$

$f_\xi(T^*T) = \langle T\xi | T\xi \rangle \geq 0 \implies f_\xi \geq 0;$

$\|f_\xi\| = \|\xi\|^2. \qquad f_\xi$ is a state $\iff \|\xi\| = 1.$

(<u>vector state</u>)

Example 5 / exer. $A = \ell^1(G)$; $G = $ group.

(1) $f : A \to \mathbb{C}$, $f(a) = a(e)$,

is a state.

(2) f extends to a state $\tilde{f}$ on $C^*_r(G)$

(3) $\tilde{f}$ is a vector state.

<u>Thm.</u> $A = C^*$-alg, $f : A \to \mathbb{C}$, $f \geq 0$. Then:

(1) $f(A_{sa}) \subset \mathbb{R}$.

(2) $f(a^*) = \overline{f(a)} \quad (a \in A)$

(3) f is continuous.

<u>Proof.</u> (1) $\forall a \in A_{sa} \quad a = a_+ - a_-$; $a_\pm \in A_{pos}$.

(2) follows from (1) (see the proof for characters)

(3) Claim: $\exists C \geq 0$ s.t. $f(a) \leq C\|a\| \ \forall a \in A_{pos}$.

If not, then $\forall n \in \mathbb{N} \ \exists a_n \in A_{pos}$ s.t. $\|a_n\| \leq 1$,

$f(a_n) \geq 4^n$.

Let $a = \sum_{n=1}^{\infty} \frac{a_n}{2^n} \implies f(a) \geq f\left(\frac{a_n}{2^n}\right) \geq 2^n \ \forall n,$

a contr. $\implies \exists\, C \geq 0$ s.t. $f(a) \leq C\|a\| \ \forall a \geq 0.$

$\forall a \in A \quad a = (b_+ - b_-) + i(c_+ - c_-)$

($b = b_+ - b_-$, $c = c_+ - c_-$ are selfadj).

$\|b_\pm\| \leq \|b\| \leq \|a\|$ (because $b = \frac{a+a^*}{2}$)

Similarly, $\|c_\pm\| \leq \|a\|$

$\Rightarrow |f(a)| \leq 4C\|a\|.$ $\square$.