Positive elements

A = C^*-alg, $a \in A$.

Def a is *positive* $(a \geq 0) \iff a = a^*$ and $\sigma_A'(a) \subset [0, +\infty)$.

(if A is unital, then $\sigma_A'(a) \subset [0, \infty) \iff \sigma_A(a) \subset [0, +\infty)$).

Example $A = C_0(X)$ (X = loc. comp. Hausd. space)

$a \in A$ $\quad a \geq 0 \iff a(x) \geq 0 \ \forall x \in X$.

Notation $A_{pos} = \{a \in A : a \geq 0\}$.

Prop. 1 (1) $\varphi : A \to B$ $*$-hom $(A, B = C^*\text{-alg})$
$\Rightarrow \varphi(A_{pos}) \subset B_{pos}$.

(2) A is unital, $a \in A$ normal, $f \in C(\sigma(a))$, $f \geq 0$
$\Rightarrow f(a) \geq 0$.

Proof (1) $\forall a \in A_{pos} \subset A_{sa}$ $\varphi(a) \in A_{sa}$.
$\sigma'_B(\varphi(a)) \subset \sigma'_A(a) \subset [0, +\infty) \Rightarrow \varphi(a) \geq 0$.

(2) Apply (1) to $\gamma_a : C(\sigma(a)) \to A$. $\square$

Prop. 2. (1) $a \in A_{pos}$, $\lambda \in \mathbb{R}$, $\lambda \geq 0 \Rightarrow \lambda a \in A_{pos}$.
(2) $A_{pos} \cap (-A_{pos}) = \{0\}$.
(3) A_{pos} is closed in A.
(4) $a, b \in A_{pos} \Rightarrow a + b \in A_{pos}$.

Thus A_{pos} is a closed convex cone in A.

Lemma. A = unital C^*-alg, $a \in A$, $\|a\| \leq 1$.
Then: $a \geq 0 \Leftrightarrow a = a^*$ and $\|1 - a\| \leq 1$.

Proof. Clear if $A = C(X)$ (X = compact space)
Gen. case: consider $C^*_A(a, 1)$ and apply the 1st G-N Thm. $\square$

Proof of Prop 2. (1) clear.

(2) $a \in A_{pos} \cap (-A_{pos}) \Rightarrow \sigma'_A(a) = \{0\} \Rightarrow a = 0$.

(3), (4) $\quad \mathbb{B} = \{a \in A : \|a\| \le 1\}$.

Lemma $\Rightarrow \mathbb{B} \cap A_{pos} = \mathbb{B} \cap A_{sa} \cap \{a : \|1_+ - a\| \le 1\}$.

$\Rightarrow \mathbb{B} \cap A_{pos}$ is closed and convex $\Rightarrow$

$\Rightarrow$ (3), (4). $\square$

<u>Notation</u> $\forall a \in A_{pos}$ $\sqrt{a} = f(a)$, where $f(t) = \sqrt{t}$ $(t \in [0, +\infty))$.

<u>Prop 3.</u> $\sqrt{a}$ is a unique positive elem. of A s.t. $(\sqrt{a})^2 = a$.

$$\boxed{g(f(a)) = (g \circ f)(a)}$$

<u>Superposition property</u>

<u>Proof.</u> $\sqrt{a} \ge 0$ by Prop. 1

Superpos. prop. $\Rightarrow (\sqrt{a})^2 = a$.

Suppose $b \ge 0$, $b^2 = a$. Superpos. prop $\Rightarrow$

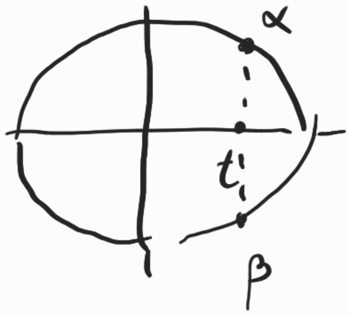
$\Rightarrow b = \sqrt{b^2} = \sqrt{a}$. $\square$

<u>Exer. 1.</u> H = Hilb. space. $T \in \mathcal{B}(H)$

$T \ge 0 \iff \langle Tx | x \rangle \ge 0 \ \forall x \in H$.

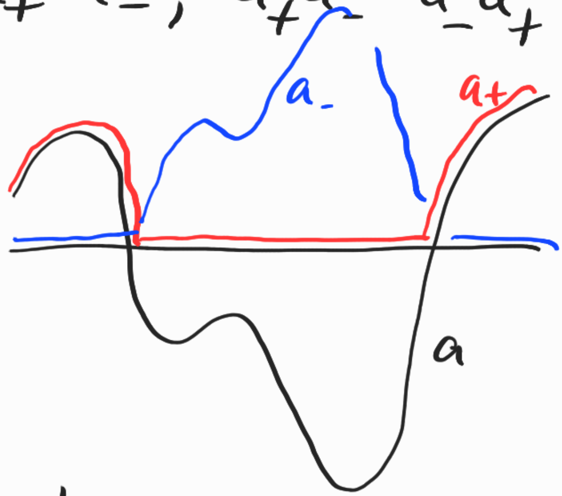
<u>Exer. 2.</u> A = unital C^*-alg $\Rightarrow$ every $a \in A$ is a lin. comb. of 4 unitaries.

<u>Hint</u>:



$$t = \frac{\alpha + \beta}{2}$$

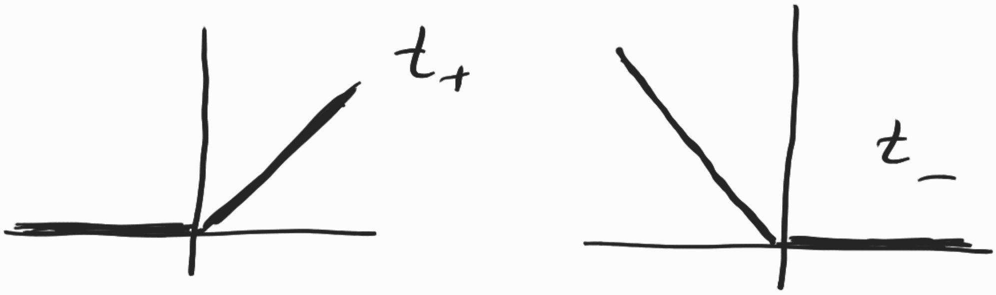
Prop. 4. $\forall a \in A_{sa}$ $\exists$ a unique pair (a_+, a_-) of positive elements s.t. $a = a_+ - a_-$, $a_+ a_- = a_- a_+ = 0$.



Proof

Consider

$t_+, t_- : \mathbb{R} \to [0, +\infty)$



$a_{\pm} = t_{\pm}(a)$ are what we need.

Suppose $a_1, a_2 \in A_{pos}$, $a = a_1 - a_2$, $a_1 a_2 = a_2 a_1 = 0$.

Let $|a| = a_+ + a_- = |t|(a)$

$$\left.\begin{array}{l} a^2 = (a_1 - a_2)^2 = a_1^2 + a_2^2 = (a_1 + a_2)^2 \\ \quad \| \\ (a_+ + a_-)^2 = |a|^2 \end{array}\right\} \overset{\sqrt{\ }}{\Longrightarrow} a_1 + a_2 = |a|$$

$$\Rightarrow a_1 = \frac{|a| + a}{2} = a_+ \ ; \quad a_2 = \frac{|a| - a}{2} = a_- . \quad \square$$

Thm. (I. Kaplansky) (1960ies).

$\forall a \in A \quad a^* a \geq 0.$

Lemma 1. $\forall x \in A \quad x^* x + x x^* \geq 0$

Proof. $x = y + iz \quad (y, z \in A_{sa}) \Longrightarrow$

$x^* = y - iz.$

$$\Rightarrow x^*x = y^2 + z^2 + i(yz - zy)$$
$$xx^* = y^2 + z^2 + i(zy - yz)$$
$$\Rightarrow x^*x + xx^* = 2(y^2 + z^2) \geq 0. \quad \text{(see Props 1, 2.)} \quad \square$$

Lemma 2. A = unital alg, $a, b \in A \Rightarrow$
$$\Rightarrow \sigma(ab) \cup \{0\} = \sigma(ba) \cup \{0\}.$$

Proof. It suff. to show that
$$1 - ab \in A^\times \Leftrightarrow 1 - ba \in A^\times$$
(because $\forall \lambda \neq 0 \quad ab - \lambda 1 = -\lambda(1 - \lambda^{-1} ab)$)

If $1 - ab \in A^\times$, $c = (1 - ab)^{-1} \Rightarrow 1 + bca$ is the inverse of $1 - ba$. $\square$.

Informal explanation:
$$(1 - ab)^{-1} = 1 + ab + (ab)^2 + \dots =$$
$$= 1 + ab + abab + ababab + \dots$$
$$\Rightarrow (1 - ba)^{-1} = 1 + ba + baba + bababa + \dots$$
$$= 1 + b(1 + ab + abab + \dots)a$$

Proof of Thm

Let $b = a^*a$; $b = b_+ - b_-$ $(b_\pm \geq 0, b_+ b_- = b_- b_+ = 0)$

We want: $b_- = 0$. Let $x = ab_-$.
$$x^*x = b_- a^* a b_- = b_-(b_+ - b_-)b_- = -b_-^3$$
$$\Rightarrow -x^*x \geq 0.$$
$$xx^* = \underbrace{(x^*x + xx^*)}_{\geq 0} + \underbrace{(-x^*x)}_{\geq 0} \geq 0 \xRightarrow{L2} x^*x \geq 0$$

$\Rightarrow x^*x = 0 \Rightarrow b_-^3 = 0 \xRightarrow[\sqrt[3]{\ }]{} b_- = 0.$ $\square$.

<u>Notation.</u> $\forall a \in A$ $|a| = \sqrt{a^*a} \in A_{pos}$.

<u>Observe</u>: if $a = a^*$, then $|a| = \sqrt{a^2} = |t|(a) =$
$= a_+ + a_-$.

<u>Cor.</u> $a \in A$. TFAE:

(1) $a \geqslant 0$

(2) $\exists\, b \geqslant 0$ s.t. $a = b^2$;

(3) $\exists\, b \in A_{sa}$ s.t. $a = b^2$;

(4) $\exists\, b \in A$ s.t. $a = b^*b$.

<u>Def</u> $a, b \in A_{sa}$. $a \leq b \Leftrightarrow b - a \geqslant 0$.

<u>Prop.</u> (1) $(A_{sa}, \leq)$ is a poset.

(2) $a \leq b$, $\lambda \in \mathbb{R}$, $\lambda \geqslant 0 \Rightarrow \lambda a \leq \lambda b$

(3) $a_1 \leq b_1$, $a_2 \leq b_2 \Rightarrow a_1 + a_2 \leq b_1 + b_2$.

(4) $a \leq b$, $x \in A \Rightarrow x^*ax \leq x^*bx$.

(5) A is unital, $a \in A$, $a \geqslant 1 \Rightarrow a^{-1} \leq 1$.

(6) A is unital, $0 \leq a \leq b$, $a, b \in A^\times \Rightarrow b^{-1} \leq a^{-1}$!

(7) $0 \leq a \leq b \Rightarrow \|a\| \leq \|b\|$.

(8) $a \in A_{pos}$. $\|a\| \leq 1 \Leftrightarrow a \leq 1_+$
$\Leftrightarrow$ (if A is unital) $a \leq 1$.

Proof (1)-(3): see Prop. 2.

(4) $b - a = c^*c$, $c \in A$

$x^*bx - x^*ax = x^*c^*cx = (cx)^*(cx) \geq 0$

(5) Apply (4) to $x = a^{-1/2}$.

(6) $0 \leq a \leq b \overset{}{\underset{(4)}{\Longrightarrow}} 1 \leq a^{-1/2} b a^{-1/2} \underset{(5)}{\Longrightarrow} a^{\frac{1}{2}} b^{-1} a^{\frac{1}{2}} \leq 1$

$\underset{(4)}{\Longrightarrow} b^{-1} \leq a^{-1}$.

(7) Apply the 1st G-N Thm to $C^*_{A_+}(b, 1_+) \Longrightarrow$

$\Rightarrow b \leq \|b\| 1_+ \Longrightarrow a \leq \|b\| 1_+ \Longrightarrow$

(1GN to $C^*_{A_+}(a, 1)$) $\Rightarrow \|a\| \leq \|b\|$

(8) Apply the 1GN to $C^*_{A_+}(a, 1_+)$ or $C^*_A(a, 1)$. $\square$

Approximate identities

$(\Lambda, \leq)$ poset.

Def. Λ is directed $\Longleftrightarrow \forall \lambda, \mu \in \Lambda \; \exists \nu \in \Lambda$ s.t. $\lambda \leq \nu$, $\mu \leq \nu$.

Examples (1) $(\mathbb{N}, \leq)$ $\leq$ is the standard order.

(2) $\Lambda = (\{\text{neighborhoods of } x \in X\}; \supset)$.

(X = top. space)

X = top. space

Def A net in X is a map $x : \Lambda \to X$ (where Λ is a directed poset)

Notation $x = (x_\lambda)_{\lambda \in \Lambda}$ $\quad x(\lambda) = x_\lambda$.

Def A net $(x_\lambda)_{\lambda\in\Lambda}$ converges to $x \in X$ ($x_\lambda \to x$ or $\lim\limits_{\Lambda} x_\lambda = x$) $\iff$ $\forall$ nbhd $U \ni x$ $\exists \lambda_0 \in \Lambda$ s.t. $\forall \lambda \geq \lambda_0$ $x_\lambda \in U$.

Example. Λ = the poset from Ex (2). $\forall U \in \Lambda$ choose any $x_U \in U$. Then $x_U \to x$.

A = normed algebra

Def An approximate identity (a.i) in A is a net $(e_\lambda)_{\lambda\in\Lambda}$ in A s.t. $\forall a \in A$ $ae_\lambda \to a$, $e_\lambda a \to a$.

Def (1) An a.i. $(e_\lambda)_{\lambda\in\Lambda}$ is sequential $\iff$ $\iff$ $\Lambda = \mathbb{N}$ with the usual order.

(2) An a.i. (e_λ) is bounded $\iff$ $\exists C \geq 0$ s.t. $\|e_\lambda\| \leq C$ $\forall \lambda$.

Example 1. $A = c_0 = C_0(\mathbb{N}) = \{a = (a_n) \in \mathbb{C}^{\mathbb{N}} : a_n \to 0\}$

$\forall n \in \mathbb{N} \quad e_n = (\underbrace{1, \dots, 1}_{n}, 0, 0, \dots) \in A$

$\Rightarrow (e_n)$ is a b.a.i. in A. Indeed: $\forall a \in A$

$$\|a - ae_n\| = \sup_{k>n} |a_k| \to 0 \quad (n \to \infty)$$

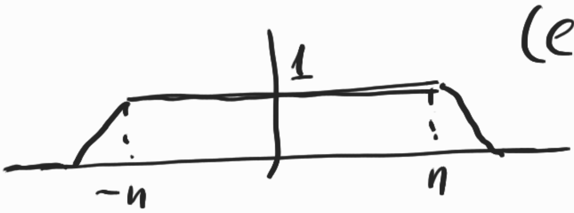
Example 2. $A = \ell^1$ with pointwise mult.

$\Rightarrow (e_n)$ from Ex 1 is an unbdd a.i. in ℓ^1.

Exer. ℓ^1 does not have a b.a.i.

Example 3. $A = C_0(\mathbb{R})$

$(e_n)_{n \in \mathbb{N}}$ is a b.a.i. (exer.)



Example 4. $A = C_0(X)$ (X = loc. comp. Hausd. space)

$\Lambda = \{\text{comp. sets } K \subset X\}$. $(\Lambda, \subset)$ is a dir. poset.

$\forall K \in \Lambda$ choose $e_K \in C_0(X)$ s.t.

$e_K|_K = 1$, $\|e_K\| \le 1 \underset{\text{(exer)}}{\Longrightarrow} (e_K)_{K \in \Lambda}$ is a b.a.i

Exer. $C_0(X)$ has a sequential a.i. $\Longleftrightarrow$

$\Longleftrightarrow X$ is σ-compact.

Example 5 $A = \mathcal{K}(H)$ H = Hilb. space

$\Lambda = \{$fin-dim. subspaces $L \subset H\}$.

$(\Lambda, \subset)$ is a dir. poset.

$\forall\, L \in \Lambda$ let P_L = the orth. proj. onto L.

$\underset{\text{(exer)}}{\Longrightarrow}$ $(P_L)_{L \in \Lambda}$ is a b.a.i. in $\mathcal{K}(H)$

Exer. $\mathcal{K}(H)$ has a seq. a.i. $\Longleftrightarrow$ H is separable.

Example 6. (1) $(A$, zero mult.$)$ does not have an a.i.

(2) $A = \{ f \in C^1[0,1] : f(0) = 0 \}$ does not have an a.i. (exer.)