The 1st Gelfand-Naimark Thm.

Thm 1. $A = C^*$-alg, $a \in A$ normal $(aa^* = a^*a)$
Then $r(a) = \|a\|$.

Thm 2. $A = C^*$-alg, $a \in A$ selfadj. $(a^* = a)$
Then $\sigma'_A(a) \subset \mathbb{R}$.

Notation $A_{sa} = \{a \in A : a^* = a\}$.

Cor. 1 $A = C^*$-alg, $\chi : A \to \mathbb{C}$ char. $\Longrightarrow$
$\Rightarrow \chi$ is a $*$-char.

Proof $\forall a \in A_{sa}$ $\sigma'_A(a) \subset \mathbb{R} \Rightarrow \sigma'_{\mathbb{C}}(\chi(a)) \subset \mathbb{R}$,
that is, $\chi(a) \in \mathbb{R}$.

$\forall a \in A$ $a = b + ic$ $(b, c \in A_{sa}) \Rightarrow \chi(b), \chi(c) \in \mathbb{R}$

$$\chi(a^*) = \chi(b - ic) = \chi(b) - i\chi(c) = \overline{\chi(b) + i\chi(c)} =$$
$$= \overline{\chi(a)}. \quad \square$$

Cor. 2. A = unital C^*-alg, $B \subset A$ closed $*$-subalg
s.t. $1_A \in B$. Then B is spec. invariant in A
(that is, $\sigma_B(b) = \sigma_A(b)$ $\forall b \in B$)
$(\Longleftrightarrow B \cap A^\times = B^\times)$.

Proof. Let $b \in B \cap A^\times \Rightarrow b^* b \in B_{sa} \cap A^\times$

$\Rightarrow \sigma_B(b^*b) \subset \mathbb{R} \Rightarrow \forall t \in \mathbb{R} \setminus \{0\}$ $b^*b + it1 \in B^\times$.

$$\underbrace{(b^*b + it1)^{-1}}_{\in B} \longrightarrow (b^*b)^{-1} \quad (t \to 0)$$

$\Rightarrow (b^*b)^{-1} \in B$; $\underbrace{(b^*b)^{-1} b^*}_{\in B} b = 1 \Rightarrow$

$\Rightarrow b$ is left invertible in B.

A similar arg. applied to bb^* shows that b is right invertible in B $\underset{\text{(exer)}}{\Longrightarrow} b \in B^\times$. $\square$

Thm (Stone-Weierstrass thm).

X = loc. comp. Hausd. top. space, $A \subset C_0(X)$ a $*$-subalgebra. Suppose that

(1) A separates the points of X;

(2) $\forall x \in X \ \exists f \in A \ f(x) \neq 0$.

Then A is dense in $C_0(X)$

Thm (Gelfand, Naimark).

A = comm. C^*-alg. Then the Gelfand transf. $\Gamma_A : A \longrightarrow C_0(\mathrm{Max}\,A)$ is an isometric $*$-isomorphism.

Proof $\forall a \in A \ \forall x \in \operatorname{Max} A \cong \hat{A}$

$\Gamma(a^*)(x) = x(a^*) = \overline{x(a)} = \overline{\Gamma(a)(x)} \implies$

$\implies \Gamma$ is a $*$-hom. $\implies \Gamma(A) \subset C_0(\operatorname{Max} A)$ is a $*$-subalg.

Stone-Weir: $\implies \Gamma(A)$ is dense in $C_0(\operatorname{Max} A)$

$\forall a \in A_{sa} \quad \|\Gamma(a)\| = r(a) = \|a\|$

$\forall a \in A \quad \|\Gamma(a)\|^2 = \|\Gamma(a)^* \Gamma(a)\| = \|\Gamma(a^* a)\| =$

$= \|a^* a\| = \|a\|^2 \implies \Gamma$ is isometric

$\implies \Gamma$ is an isometric $*$-isomorphism $\square$

Def. $\mathcal{A}, \mathcal{B}$ = categories.

A covariant functor $F: \mathcal{A} \to \mathcal{B}$ is an <u>equivalence</u> $\iff \exists$ a cov. functor $G: \mathcal{B} \to \mathcal{A}$ s.t. $G \circ F \cong 1_{\mathcal{A}}$, $F \circ G \cong 1_{\mathcal{B}}$.

G is a <u>quasi-inverse</u> of F.

Notation CUC^*

objects: comm. unital C^*-algebras

morphisms: (cont.) unital $*$-homomorphisms.

Prop. $\mathrm{Comp}^{op} \underset{\mathrm{Max}}{\overset{C}{\rightleftarrows}} \mathrm{CUC}^*$

are equivalences of categories;

$C \circ \mathrm{Max} \underset{\Gamma}{\cong} 1_{\mathrm{CUC}^*}$;

$\mathrm{Max} \circ C \underset{\varepsilon}{\cong} 1_{\mathrm{Comp}^{op}}$

Notation. A = C^*-alg, $S \subset A$.

$C^*_A(S) = \bigcap \{ B \subset A : B \text{ is a } C^*\text{-subalg}, B \supset S \}$

is the C^*-subalg of A generated by S.

Observe:

(1) $C^*_A(S) = \overline{\mathrm{span}}\{ a_1 \dots a_n : a_i \in S \cup S^*, n \in \mathbb{N} \}$.

(2) A is unital, $a \in A$ normal

$\Rightarrow C^*_A(a,1) = \overline{\{ p(a,a^*) : p \in \mathbb{C}[x,y] \}}$.

If $a = a^*$, then

$C^*_A(a,1) = \overline{\{ p(a) : p \in \mathbb{C}[x] \}}$.

Prop. A, B = C^*-alg, $\varphi : A \to B$ injective $*$-hom

$\Rightarrow \varphi$ is isometric.

Proof. We may assume that A, B unital,

$\varphi(1_A) = 1_B$.

It suff. to show that $\|\varphi(a)\| = \|a\|$ $\forall a \in A_{sa}$.
(see the pf of the G-N thm)

$\forall a \in A_{sa}$ let $A_1 = C^*_A(a,1)$, $B_1 = C^*_B(\varphi(a), 1)$

We want: $\varphi : A_1 \longrightarrow B_1$ is isometric.

$$\begin{array}{ccc} A_1 & \longrightarrow & B_1 \\ \wr\| & & \wr\| \\ C(X) & & C(Y) \end{array}$$

We have $\varphi = f^\bullet$, where $f : Y \to X$ is cont.

φ is inj. $\Longrightarrow$ f is surj.

$$\|\varphi(a)\| = \|a \circ f\| = \sup_{y \in Y} |a(f(y))| = \sup_{x \in X} |a(x)| = \|a\|$$

$\square$

Continuous functional calculus

A = unital Banach $*$-alg, $a \in A$ normal

$K \subset \mathbb{C}$ compact; $t : K \hookrightarrow \mathbb{C}$ inclusion map.

<u>Def.</u> A <u>continuous functional calculus</u> for a on K is a cont. $\underbrace{\ast}_{\text{unital}}$-hom. $\gamma_a : C(K) \longrightarrow A$ s.t. $\gamma_a(t) = a$.

Prop. If $\gamma_a \exists$, then it is unique.

Proof. St-Weier. $\Rightarrow$ the subalg of $C(K)$ gener by $1, t, \bar{t}$ is dense in $C(K)$ $\square$

Prop. If $\gamma_a \exists$, then $\sigma(a) \subset K$.

Proof. $\sigma(a) = \sigma(\gamma_a(t)) \subset \sigma_{C(K)}(t) = K$ $\square$.

Exer. Construct a unital Ban. $*$-alg A and a normal el. $a \in A$ s.t. γ_a never exists.

Thm. A = unital C^*-alg, $a \in A$ normal $\Rightarrow \forall$ compact $K \subset \mathbb{C}$ s.t. $K \supset \sigma(a)$ $\exists$ a unique cont. func. calc. $\gamma_a : C(K) \to A$

Moreover, if $K = \sigma(a)$, then γ_a is an isometric $*$-isom onto $C^*_A(a, 1)$.

Proof. We may assume that $K = \sigma(a)$:

$$\begin{tikzcd} C(K) \arrow[r, "restr."] \arrow[rr, bend right, "\gamma_a"'] & C(\sigma(a)) \arrow[r, "\gamma_a"] & A \end{tikzcd}$$

Let $B = C^*_A(a, 1)$. B is a comm. C^*-alg;

$\sigma_B(a) = \sigma_A(a)$ (Spec. Invariance)

Let $\varphi: \mathrm{Max}B \to \sigma(a)$, $\varphi(x) = \hat{a}(x)$.

By the properties of $\hat{a}$, φ is surjective.

If $x_1, x_2 \in \mathrm{Max}B$, $\varphi(x_1) = \varphi(x_2) \Rightarrow$

$\Rightarrow x_1(a) = x_2(a) \Rightarrow x_1(a^*) = x_2(a^*) \Rightarrow x_1 = x_2$.

$\Rightarrow \varphi$ is bijective $\Rightarrow \varphi$ is a homeo. (by comp.)

$$\begin{tikzcd}
C(\sigma(a)) \arrow[r, "\varphi^\circ", "\sim"'] \arrow[rrr, bend right, "\gamma_a"'] & C(\mathrm{Max}B) \arrow[r, "\Gamma_B^{-1}", "\sim"'] & B \arrow[r, hook] & A
\end{tikzcd}$$

$\gamma_a(t) \overset{?}{=} a \iff \Gamma_B^{-1}(\varphi^\circ(t)) \overset{?}{=} a \iff \underset{t\circ\varphi}{\underset{\parallel}{\varphi^\circ(t)}} \overset{?}{=} \hat{a}$ Ok.

$\square$

<u>Observe</u> (1) $f \in \mathbb{C}[x] \Rightarrow \gamma_a(f|_K) = f(a)$.

(2) $f \in \mathbb{C}[x,y]$. Define $\tilde{f}: \mathbb{C} \to \mathbb{C}$ by $\tilde{f}(z) = f(z, \bar{z})$.

Then $\gamma_a(\tilde{f}|_K) = f(a, a^*) = \tilde{f}(a)$.

<u>Notation</u>. $\forall f \in C(\sigma(a))$ $f(a) = \gamma_a(f)$.

<u>Thm.</u> (<u>Spectral Mapping Thm</u>)

A = unital C^*-alg, $a \in A$ normal.

$\forall f \in C(\sigma(a))$ $\sigma(f(a)) = f(\sigma(a))$.

Proof. $\sigma_A(f(a)) = \sigma_A(\gamma_a(f)) = \sigma_B(\gamma_a(f)) =$

(where $B = C^*_A(a,1)$) $= \sigma_{C(\sigma(a))}(f)$

$= f(\sigma(a))$. $\square$

Thm (Superposition property).

$f \in C(\sigma(a))$, $g \in C(\sigma(f(a))) \Rightarrow$

$\Rightarrow g(f(a)) = (g \circ f)(a)$

Proof Consider $\gamma : C(\sigma(f(a))) \longrightarrow A$,

$\gamma(g) = (g \circ f)(a)$.

γ is a unital $*$-hom, $\gamma(t) = (t \circ f)(a) = f(a)$.

$\Rightarrow \gamma = \gamma_{f(a)}$. $\square$

Example/exer 1. $A = C(X)$; $a \in A$; $f \in C(\sigma(a))$

$\Rightarrow f(a) = f \circ a$.

Example/exer 2. $(X, \mu) =$ σ-fin. measure space,

$\varphi \in L^\infty(X,\mu)$; $M_\varphi \in \mathcal{B}(L^2(X,\mu))$, $\psi \mapsto \varphi\psi$.

$\Rightarrow \forall f \in C(\sigma(M_\varphi))$ $f(M_\varphi) = M_{f \circ \varphi}$.

Nonunital case

$A = C^*$-alg, $a \in A$ normal; $A \subset A_+$.

$f \in C(\sigma'_A(a))$ $\quad f(a) \in A_+$. $\qquad 0 \in \sigma'_A(a)$

Exer. $f(a) \in A \iff f(0) = 0$.

$$\begin{array}{ccc} C(\sigma'_A(a)) & \xrightarrow{\gamma} & A_+ \\ \cup & & \cup \\ I_0 & \xrightarrow{\gamma} & A \end{array}$$