# $C^*$-algebras

**Def.** $A$ = algebra. An *involution* on $A$ is
$A \to A$, $a \in A \mapsto a^* \in A$ s.t.

(1) $a^{**} = a \quad (a \in A)$;

(2) $(\lambda a + \mu b)^* = \bar{\lambda} a^* + \bar{\mu} b^* \quad (\lambda, \mu \in \mathbb{C}, a, b \in A)$

(3) $(ab)^* = b^* a^* \quad (a, b \in A)$

$(A, *)$ is a *$*$-algebra*

**Def.** A *Banach $*$-alg* is a Ban. alg. equipped with an invol. s.t. $\|a^*\| = \|a\| \quad (a \in A)$

**Def.** A Ban. $*$-alg $A$ is a *$C^*$-algebra* if
$\forall a \in A \quad \|a^* a\| = \|a\|^2$.
( *$C^*$-axiom*, *$C^*$-identity*)

**Def.** $A, B$ $*$-algebras.
An alg. hom $\varphi: A \to B$ is a *$*$-homomorphism* if $\varphi(a^*) = \varphi(a)^* \quad (a \in A)$.

**Def.** $A$ = $*$-alg, $B \subset A$ subalg.
$B$ is a *$*$-subalgebra* $\iff \forall b \in B \quad b^* \in B$.

Example $0$, $\mathbb{C}$ are $C^*$-alg; $\lambda^* = \bar{\lambda}$ $(\lambda \in \mathbb{C})$.

Example $X$ = a set $\Rightarrow$ $\ell^\infty(X)$ is a $C^*$-alg,

$f^*(x) = \overline{f(x)}$ $(x \in X)$.

Example $X$ = top. space $\Rightarrow$

$\Rightarrow$ $C_0(X) \subset C_b(X) \subset \ell^\infty(X)$ are closed $*$-subalgebras $\Rightarrow$ $C^*$-algebras.

Example. $(X,\mu)$ = meas. space $\Rightarrow$ $L^\infty(X,\mu)$ is a $C^*$-alg w.r.t. $f^*(x) = \overline{f(x)}$.

Example $H$ = Hilb. space $\Rightarrow$ $\mathcal{B}(H)$ is a $C^*$-alg.

$\forall\, T \in \mathcal{B}(H)$ the adjoint $T^* \in \mathcal{B}(H)$ is uniquely determ. by $\langle T^* x | y \rangle = \langle x | Ty \rangle$ $(x, y \in H)$.

Def. $A$ = $*$-alg, $H$ = Hilb. space.

A *$*$-representation* of $A$ on $H$ is a $*$-hom

$\pi : A \to \mathcal{B}(H)$.

$\pi$ is *faithful* $\Leftrightarrow$ $\operatorname{Ker} \pi = 0$. (точное; верное)

Example. $\mathcal{K}(H) \subset \mathcal{B}(H)$ is a closed 2-sided $*$-ideal $\Rightarrow$ $\mathcal{K}(H)$ is a $C^*$-alg.

Example/exer. $C^n[a,b]$ is a Ban. $*$-alg w.r.t. $f^*(x)=\overline{f(x)}$, but is not a $C^*$-alg (unless $n=0$)

Example/exer $\mathcal{A}(\overline{\mathbb{D}})$ is a Ban. $*$-alg w.r.t. $f^*(z)=\overline{f(\overline{z})}$ $(z\in\overline{\mathbb{D}})$, but is not a $C^*$-alg.

Example/exer. $G$=group $\Rightarrow \ell^1(G)$ is a Ban. $*$-alg w.r.t. $f^*(x)=\overline{f(x^{-1})}$, but is not a $C^*$-alg (unless $G=\{e\}$)

Example (reduced group $C^*$-alg)

$G$ = group; $\ell^2(G)=\{f: G\to\mathbb{C} : \sum_{x\in G}|f(x)|^2<\infty\}$

$\ell^2(G)$ is a Hilb. space; $\langle f|g\rangle=\sum_{x\in G} f(x)\overline{g(x)}$.

Exer. (1) $\forall f\in\ell^1(G), g\in\ell^2(G)$

$f*g$ is defined on $G$, $f*g\in\ell^2(G)$,

$\|f*g\|_2\le\|f\|_1\|g\|_2$.

(2) The map $\lambda:\ell^1(G)\to\mathcal{B}(\ell^2(G))$, $\lambda(f)g=f*g$ is a faithful $*$-rep. (left regular rep.)

**Def** The reduced $C^*$-alg of $G$ is
$$C^*_r(G) = \overline{\mathrm{Im}\,\lambda} \subset \mathcal{B}(\ell^2(G)).$$
$$\ell^1(G) \xrightarrow{\lambda} C^*_r(G).$$

## Products and unitizations of $C^*$-algebras

### Product

Observe:

(1) $A, B = *$-alg $\Rightarrow A\times B \; (= A\oplus B)$ is a $*$-alg w.r.t. $(a,b)^* = (a^*, b^*)$.

(2) $A, B =$ Ban. $*$-alg $\Rightarrow A\oplus B$ is a Ban. $*$-alg w.r.t. $\|(a,b)\| = \max\{\|a\|, \|b\|\}$.

(3) $A, B = C^*$-alg $\Rightarrow$ so is $A\oplus B$.

### Unitization

Observe: $A =$ Ban. $*$-alg $\Rightarrow$ so is $A_+$:
$$(a+\lambda 1_+)^* = a^* + \bar{\lambda} 1_+ \quad (a\in A, \lambda\in\mathbb{C});$$
$$\|a+\lambda 1_+\| = \|a\| + |\lambda|. \qquad (1)$$

Exer. $A = C^*$-alg, $A \neq 0 \implies$ norm (1) does not satisfy the $C^*$-axiom.

$A$ = unital $C^*$-alg $\Rightarrow A_+ = A \oplus \mathbb{C}(1_+ - 1_A) \cong$
$\cong A \oplus \mathbb{C}$ as $*$-algebras $\Rightarrow$
$\Rightarrow A_+$ is a $C^*$-alg w.r.t.
$\|a + \lambda(1_A - 1_+)\| = \max\{\|a\|, |\lambda|\}$. Equivalently:
$\|a + \lambda 1_+\| = \max\{\|a + \lambda 1_A\|, |\lambda|\}$.

Prop. $A$ = nonunital (strictly) $C^*$-alg
$\forall a \in A_+$ let $L_a : A \to A$, $L_a(b) = ab$;
let $\|a\|_+ = \|L_a\| = \sup\{\|ab\| : b \in A, \|b\| \le 1\}$.
Then: (1) $\|\cdot\|_+$ is a norm on $A_+$;
(2) $\forall a \in A$ $\|a\| = \|a\|_+$;
(3) $(A_+, \|\cdot\|_+)$ is a $C^*$-algebra.

Lemma 1 / exer. $E$ = normed sp, $E_0 \subset E$ vec. subsp, $\operatorname{codim}_E E_0 = 1$. If $E_0$ is a Ban. space, then so is $E$.

<u>Lemma 2/exer.</u> $A$ = Ban. alg equipped with an involution s.t. $\|a\|^2 \le \|a^*a\|$ $\forall a \in A$
$\Rightarrow$ $A$ is a $C^*$-alg.

<u>Proof of Prop</u>

(2) $\forall a, b \in A$ $\|ab\| \le \|a\|\|b\|$ $\Rightarrow$

$\Rightarrow \|a\|_+ \le \|a\|$;

$\|aa^*\| = \|a\|^2 = \|a\| \cdot \|a^*\| \Rightarrow \|a\|_+ = \|a\|$

(1) Clearly, $\|\cdot\|_+$ is a seminorm on $A_+$.

Suppose $a \in A_+$, $a \ne 0$, $\|a\|_+ = 0$.

$a = b + \lambda 1_+$ $(b \in A, \lambda \in \mathbb{C})$. By (2), $\lambda \ne 0$.

We have $L_a = 0$, that is, $\forall c \in A$

$0 = ac = bc + \lambda c \Rightarrow (-\lambda^{-1} b) c = c \Rightarrow$

$\Rightarrow e = -\lambda^{-1} b$ is a left identity in $A$ $\Rightarrow$

$\Rightarrow e^*$ is a right identity $\underset{\text{(exer)}}{\Longrightarrow}$ $A$ is unital, a contr. $\Rightarrow \|\cdot\|_+$ is a norm on $A_+$.

(3) Lemma 1 $\Rightarrow$ $A_+$ is a Banach alg w.r.t. $\|\cdot\|_+$.

$\forall a \in A_+, b \in A$

$$\|ab\|^2 = \|(ab)^*ab\| = \|b^*a^*ab\| \leq$$
$$\leq \|b^*\|\,\|a^*ab\| \leq \|b^*\|\,\|a^*a\|_+\|b\| =$$
$$= \|a^*a\|_+\|b\|^2. \implies \|a\|_+^2 \leq \|a^*a\|_+$$

$\implies A_+$ is a $C^*$-alg. $\square$

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## Spectral properties of $C^*$-algebras. The 1st (commutative) Gelfand-Naimark theorem

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$A = *$-alg.

<u>Def.</u> $a \in A$ is <u>selfadjoint</u> $\iff a^* = a$.
(<u>hermitian</u>)

$a \in A$ is <u>normal</u> $\iff aa^* = a^*a$.

If $A$ is unital, then $u \in A$ is <u>unitary</u> $\iff$
$u \in A^\times$ and $u^{-1} = u^*$.

<u>Observe</u>: (1) selfadj $\implies$ normal
unitary $\implies$ normal

(2) $\forall\, a \in A$ $a^*a$ is selfadj.

**Example 1** $A = \mathbb{C}^X$, or $A = C_b(X)$

(1) $f \in A$ is selfadj $\Longleftrightarrow$ $f(x) \in \mathbb{R}$ $\forall x \in X$.

(2) $f \in A$ is unitary $\Longleftrightarrow$ $|f(x)| = 1$ $\forall x \in X$.

**Example 2/exer.** $A = \mathcal{B}(H)$, $H$ = Hilb. space.

(1) $T \in \mathcal{B}(H)$ is selfadj $\Longleftrightarrow$ $\langle Tx|x\rangle \in \mathbb{R}$ $\forall x \in H$.

(2) $U \in \mathcal{B}(H)$ is unitary $\Longleftrightarrow$ $U$ is bijective, and $\langle Ux|Uy\rangle = \langle x|y\rangle$ $(x, y \in H)$

**Prop** $\forall a \in A$ $\exists$ a unique pair $(b, c)$ of selfadj. elements s.t. $a = b + ic$.

**Proof** $b = \dfrac{a + a^*}{2}$, $c = \dfrac{a - a^*}{2i}$. $\square$

**Thm 1.** $A = C^*$-alg, $a \in A$ normal $\Longrightarrow$ $\|a\| = r(a)$.

**Proof** $\forall b = b^* \in A$ $\|b^2\| = \|b\|^2$. $\Longrightarrow$

$$\Longrightarrow \underset{\|a\|^4}{\underset{\shortparallel}{\|a^*a\|^2}} = \|(a^*a)^2\| = \|a^*aa^*a\| =$$

$$= \|(a^*)^2 a^2\| = \|(a^2)^* a^2\| =$$

$$= \|a^2\|^2.$$

$\Longrightarrow \|a\|^2 = \|a^2\|$. Induction $\Longrightarrow$ $\|a^{2^n}\| = \|a\|^{2^n}$ $\forall n$.

$\Rightarrow r(a) = \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}} = \lim_{n\to\infty} \|a^{2^n}\|^{\frac{1}{2^n}} = \|a\|.$ $\square$

<u>Cor. 1.</u> $A = C^*$-alg $\Rightarrow \forall a \in A \quad \|a\| = \sqrt{r(a^*a)}$.

<u>Cor. 2.</u> $A = *$-alg $\Rightarrow \exists$ at most 1 norm on $A$ which makes $A$ into a $C^*$-algebra.

Equivalently, each $*$-isomorphism between $C^*$-algebras is isometric.

<u>Cor. 3</u> $A =$ Ban. $*$-alg, $B = C^*$-alg.

Each $*$-hom. $\varphi: A \to B$ is cont, and $\|\varphi\| \le 1$.

<u>Proof</u> Let $a = a^* \in A \Rightarrow \varphi(a)$ is selfadj.

$\Rightarrow \|\varphi(a)\| = r(\varphi(a)) \le r(a) \le \|a\|$

$\forall a \in A \quad a^*a$ is selfadj. $\Rightarrow$

$\Rightarrow \|\varphi(a)\|^2 = \|\varphi(a)^*\varphi(a)\| = \|\varphi(a^*a)\| \le \|a^*a\| \le \|a\|^2$

$\square$

<u>Thm. 2.</u> $A = C^*$-alg, $a \in A$ selfadjoint $\Rightarrow$

$\Rightarrow \sigma'_A(a) \subset \mathbb{R}$

<u>Proof.</u> We may assume that $A$ is unital.

Let $\lambda \in \sigma(a)$, $\lambda = \alpha + i\beta$ $(\alpha, \beta \in \mathbb{R})$.

$\forall\, t \in \mathbb{R}$ $\quad \lambda + it \in \sigma(a + it1) \Longrightarrow$

$$\Longrightarrow |\lambda + it|^2 \leq \|a + it1\|^2 = \|(a - it1)(a + it1)\| =$$

$$= \|a^2 + t^2 1\| \leq \|a^2\| + t^2.$$

$$|\lambda+it|^2 = \alpha^2 + (\beta + t)^2 = \alpha^2 + \beta^2 + 2\beta t + t^2$$

$$\Longrightarrow \alpha^2 + \beta^2 + 2\beta t \leq \|a^2\| \quad \forall\, t \in \mathbb{R}$$

$\Longrightarrow \beta = 0$, that is, $\lambda \in \mathbb{R}$. $\quad \square$