A = comm. unital Ban. alg.

$$\hat{A} \xrightarrow{\sim} \mathrm{Max}(A) \qquad \chi \mapsto \operatorname{Ker}\chi.$$
$$\cap$$
$$(A^*, wk^*)$$

$$\Gamma_A : A \longrightarrow C(\mathrm{Max}A), \quad a \in A \mapsto \hat{a}$$
$$\hat{a}(\chi) = \chi(a).$$
<u>Gelfand transform</u>

Examples: subalgebras of $C(X)$

X = compact Hausd. top. space

$\forall x \in X \quad \varepsilon_x : C(X) \to \mathbb{C}, \quad \varepsilon_x(f) = f(x)$;

$m_x = \operatorname{Ker}\varepsilon_x$.

<u>Lemma</u> $\forall$ ideal $I \subsetneq C(X)$ $\exists\, x \in X$ s.t. $I \subset m_x$.

<u>Proof.</u> Suppose $\forall x \in X$ $\exists\, f_x \in I$ s.t. $f_x(x) \neq 0$.

$\exists$ a nbhd $U_x \ni x$ s.t. $\forall y \in U_x \quad f_x(y) \neq 0$.

$X = U_{x_1} \cup \dots \cup U_{x_n}$ (by compactness)

Let $f = \sum_{i=1}^{n} |f_{x_i}|^2 = \sum_{i=1}^{n} \overline{f_{x_i}}\, f_{x_i} \in I$;

$f(y) > 0 \ \forall y \in X \Rightarrow f$ is invertible in $C(X)$

$\Rightarrow I = C(X)$, a contr. $\square$.

<u>Cor</u> The map $\varepsilon : X \to \operatorname{Max} C(X)$, $x \mapsto m_x$

$$\begin{tikzcd} X \arrow[r, "\varepsilon"] \arrow[dr] & \operatorname{Max} C(X), \; x \mapsto m_x \arrow[d, equal] \\ & \widehat{C(X)}, \; x \mapsto \varepsilon_x \end{tikzcd}$$

is a bijection.

<u>Notation.</u> X, Y compact, Hausd.

$f : X \to Y$ cont.

$f^\circ : C(Y) \to C(X)$, $\quad f^\circ(\varphi) = \varphi \circ f$.

<u>Properties of f°</u>:

(1) f° is a unital alg. hom, and $\|f^\circ\| = 1$.

(2) $(1_X)^\circ = 1_{C(X)}$

(3) $X \xrightarrow{f} Y \xrightarrow{g} Z \Rightarrow (g \circ f)^\circ = f^\circ \circ g^\circ$.

<u>Observe</u>: (1)–(3) $\Rightarrow$ if f is a homeo, then f° is an isometric isomorphism.

Thm. X = compact Hausd. top. space

$A \subset C(X)$ subalg, $1_{C(X)} \in A$. Suppose

(1) A is a Ban. alg. w.r.t a norm that dominates the sup norm.

(2) A separates the points of X

(3) $\forall\, \chi \in \hat{A}\ \exists\, x \in X$ s.t. $\chi = \varepsilon_x$.

Then the map $\varepsilon : X \to \hat{A}$, $x \mapsto \varepsilon_x$, is a homeomorphism. Moreover, the foll. diag. commutes:

$$\begin{tikzcd}
 & C(X) \\
A \arrow[ur, hook] \arrow[dr, "\Gamma_A"'] & \\
 & C(\operatorname{Max} A) \arrow[uu, "\wr"', "\varepsilon^\circ"]
\end{tikzcd}$$

Proof. (2) & (3) $\Longrightarrow$ ε is a bijection.

ε is cont $\Longleftrightarrow$ $\forall\, a \in A$ the map $x \mapsto \varepsilon(x)(a)$ is cont.

$$\varepsilon(x)(a) = a(x)$$

$a \in C(X) \Longrightarrow \varepsilon$ is cont $\Longrightarrow \varepsilon$ is a homeo.

$$(\varepsilon^\circ \Gamma)(a)(x) = \Gamma(a)(\varepsilon_x) = \varepsilon_x(a) = a(x)$$

$\Longrightarrow$ the diag. commutes. $\square$

Cor. If $A = C(X)$ (X compact, Hausd)

$\Rightarrow \Gamma_A$ is an isometric isomorphism, and $\Gamma_A^{-1} = \varepsilon^\bullet$.

Functorial properties of Γ

Comp — objects: comp. Hausd. top spaces; morphisms: cont maps.

CUBA — objects: comm. unital Ban. alg.; morphisms: cont. unital homs.

2 contravar. functors:

C: Comp $\to$ CUBA, $X \mapsto C(X)$;

$$(f : X \to Y) \mapsto \big(f^\bullet : C(Y) \to C(X),\ f^\bullet(\varphi) = \varphi \circ f\big)$$

Max: CUBA $\to$ Comp, $A \mapsto \mathrm{Max}(A)$,

$$(\varphi : A \to B) \mapsto \big(\varphi^* : \mathrm{Max}(B) \to \mathrm{Max}(A),\ \varphi^*(\chi) = \chi \circ \varphi\big).$$

φ^* is the restr. of $\varphi^* : B^* \to A^*$ (dual of φ), which is wk*-cont $\Rightarrow \varphi^* : \mathrm{Max}B \to \mathrm{Max}A$ is cont.

Exer.

(1) $\{ \varepsilon_X : X \to Max(C(X)) : X \in Comp \}$

is a natural isom. btw 1_{Comp} and $Max \circ C$.

(2) $\{ \Gamma_A : A \to C(Max(A)) : A \in CUBA \}$

is a natural transformation from 1_{CUBA} to $C \circ Max$.

(3) $\exists$ 1-1 correspondence

$$Hom_{CUBA}(A, C(X)) \cong Hom_{Comp}(X, Max A) \cong Hom_{Comp^{op}}(Max A, X)$$

$$\varphi \longmapsto \varphi^* \circ \varepsilon_X$$

$$f^\circ \circ \Gamma_A \longleftarrow f$$

Hence (Max, C) is an adjont pair of functors.

Unitization

A = algebra

$A_+ = A \oplus \mathbb{C} 1_+$ (vec. space dir. sum)

Multiplication: $(a + \lambda 1_+)(b + \mu 1_+) = ab + \lambda b + \mu a + \lambda\mu 1_+$.

A_+ becomes a unital algebra.

Def. A_+ is the <u>unitization</u> of A.

Prop 1. (exer)

A = alg, B = unital alg, $\varphi : A \to B$ alg. hom.

Define $\varphi_+ : A_+ \to B$, $\varphi_+(a + \lambda 1_+) = \varphi(a) + \lambda 1_B$.

Then: (1) φ_+ is a unital hom.

(2) $\exists$ a 1-1 correspondence

$$\mathrm{Hom}_{Alg}(A,B) \rightleftarrows \mathrm{Hom}_{Un.Alg}(A_+, B)$$

$$\varphi \longmapsto \varphi_+$$

$$\psi|_A \longleftarrow\!\shortmid\; \psi$$

Prop 2 (exer)

(1) If A is a Ban. alg, then A_+ is a Ban. alg. w.r.t. $\|a + \lambda 1_+\| = \|a\| + |\lambda|$.

(2) Prop. 1. holds if A and B are Ban. alg, and "Hom" = cont. homomorphisms.

Cor. A = Ban. alg. $\Rightarrow$ each char. $\chi : A \to \mathbb{C}$ is cont, and $\|\chi\| \le 1$.

Example / exer.

X = loc. comp. Hausd top. space
X_+ = the one-point compactification
$X_+ = X \sqcup \{\infty\}$
Topology on X_+: $\{\text{open } U \subset X\} \cup \{X_+ \setminus K : \underset{\text{comp.}}{K \subset X}\}$

Facts: (1) X_+ is a comp. Hausd. top. space
(2) Y = comp. Hausd. space, $X = Y \setminus \{y_0\}$, then X is loc. comp, and $\exists$ a homeo
$X_+ \xrightarrow{\sim} Y$; $x \in X \mapsto x$, $\infty \mapsto y_0$.

Prove: (1) $C_0(X) = \{ f|_X : f \in C(X_+),\ f(\infty) = 0 \}$

(2) $\exists$ a topol. isomorphism
$C_0(X)_+ \xrightarrow{\sim} C(X_+)$; $f + \lambda 1_+ \mapsto f + \lambda$
$(f(\infty) = 0)$

A = algebra, $a \in A$.

Def. The nonunital spectrum of a is
$\sigma_A'(a) = \sigma_{A_+}(a)$

Observe: $A \subset A_+$ is a 2-sided ideal
$\Rightarrow$ each $a \in A$ is not invertible in A_+
$\Rightarrow 0 \in \sigma'_A(a)$

Exer. (1) A_1, A_2 unital algebras, $A = A_1 \oplus A_2$.
$\Rightarrow \forall\, a = (a_1, a_2) \in A_1 \oplus A_2 \quad \sigma_A(a) = \sigma_{A_1}(a_1) \cup \sigma_{A_2}(a_2)$

(2) A = unital alg $\Rightarrow \exists$ a unital alg. isom.
$A \oplus \mathbb{C} \xrightarrow{\sim} A_+ \,, \quad (a, \lambda) \mapsto a + \lambda(1_+ - 1_A)$.

(3) A = unital alg $\Rightarrow \forall\, a \in A \quad \sigma'_A(a) = \sigma_A(a) \cup \{0\}$.

Def. A = Ban. algebra, $a \in A$.
The spectral radius of a is
$$r_A(a) = \sup\{|\lambda| : \lambda \in \sigma'_A(a)\}.$$

Thm $$r_A(a) = \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}} = \inf_{n\geq 1} \|a^n\|^{\frac{1}{n}}.$$

Max and Γ for nonunital Banach alg.

A = comm. alg.

Def. An ideal $I \subset A$ is modular (regular)
$\iff A/I$ is unital

$\iff \exists\, u \in A$ s.t. $a - au \in I \ \forall a \in A$.
(modular identity)

<u>Examples</u> (1) 0 is modular $\iff A$ is unital
$\iff$ all ideals are modular.

(2) $\chi : A \to \mathbb{C}$ character $\Rightarrow \ker\chi$ is modular.

(3) Let $A^2 = \operatorname{span}\{ab : a, b \in A\}$.
If $A^2 \neq A$, then each vec. subspace I s.t. $A^2 \subseteq I \subsetneq A$ is a non-modular ideal.
For example, $A = t\mathbb{C}[t]$, $I = tA$.

<u>Thm.</u> Each proper modular ideal of A is contained in a max. modular ideal.

<u>Proof</u>: exer.

<u>Def</u> The <u>max. spectrum</u> of A is
$\operatorname{Max}(A) = \{\text{maximal modular ideals of } A\}$.
The <u>character space</u> of A is
$$\hat{A} = \{\chi : A \to \mathbb{C} : \chi \text{ is a char}, \chi \neq 0\}.$$

<u>Notation</u> $\hat{A}_+ = \hat{A} \cup \{0 : A \to \mathbb{C}\} = \{\text{all chars}\}$.
$\operatorname{Max}_+(A) = \operatorname{Max}(A) \cup \{A\}$

Prop. (D)

$$\begin{tikzcd} \mathrm{Max}(A_+) \arrow[r, "I \mapsto I \cap A"] & \mathrm{Max}_+(A) & \mathrm{Ker}\chi \\ \widehat{A_+} \arrow[u] \arrow[r, "\chi \mapsto \chi|_A"] & \hat{A}_+ \arrow[u] & \chi \arrow[u, mapsto] \end{tikzcd}$$

The diag. is comm, and the horiz. arrows are bijections.

Proof: exer.

Prop. A = comm. Ban. algebra. Then

(1) All max. modular ideals of A are closed;

(2) All arrows in (D) are bijections.

Proof: exer.

Def. The <u>Gelfand topology</u> on $\mathrm{Max}A \cong \hat{A}$ and $\mathrm{Max}_+A \cong \hat{A}_+$ is the restriction of the weak* top. on A^*.

Prop. $\mathrm{Max}A$ and Max_+A are Hausdorff, Max_+A is compact, $\mathrm{Max}_+A \underset{\text{homeo.}}{\cong} \mathrm{Max}(A_+)$, $\mathrm{Max}A$ is loc. compact, and Max_+A is the 1-point compactification of $\mathrm{Max}A$.

Def. The <u>Gelfand transform</u> of $a\in A$ is

$$\hat{a}: \mathrm{Max}\,A \to \mathbb{C}, \quad \hat{a}(\chi)=\chi(a).$$

Prop. $\hat{a}\in C_0(\mathrm{Max}\,A)$.

Proof: continuity: clear (see the un. case);

$\infty\in \mathrm{Max}(A)_+ = 0\in \hat{A}_+$ (zero character)

$\hat{a}(0)=0 \Rightarrow \hat{a}\in C_0(\mathrm{Max}\,A)$. $\square$

Def The <u>Gelfand transform</u> of A is

$$\Gamma_A : A\to C_0(\mathrm{Max}\,A), \quad a\mapsto \hat{a}.$$

$$\begin{tikzcd}
A \arrow[r, "\Gamma_A"] \arrow[d, phantom, "\cap"] & C_0(\mathrm{Max}\,A) \arrow[d, phantom, "\cap"] \\
A_+ \arrow[r, "\Gamma_{A_+}"] & C(\mathrm{Max}\,A_+) \cong C(\mathrm{Max}_+ A)
\end{tikzcd}$$

Thm. A = comm. Ban. algebra. Then

(1) Γ_A is an alg. hom.

(2) $\|\Gamma_A\|\le 1$.

(3) $\|\hat{a}\|_\infty = r_A(a)\ \forall a\in A$

(4) $\sigma'_A(a) = \hat{a}(\mathrm{Max}\,A)\cup\{0\}\ \forall a\in A$

(5) $\mathrm{Ker}\,\Gamma_A = \bigcap\{$max $^{\text{modular}}$ ideals of $A\} = \{$quasinilp.$\}$