Spectral radius

A = unital Ban. alg, $a \in A$ $\qquad (A \neq 0)$

<u>Def</u> The <u>spectral radius</u> of a is

$$r_A(a) = \sup\{|\lambda| : \lambda \in \sigma_A(a)\}.$$

Gelfand's thm $\Rightarrow r_A(a) \leq \|a\|$.

<u>Example</u> $A = \ell^\infty(X) \Rightarrow r_A(a) = \|a\|$.

The same holds for $C_b(X)$, X = top. space.

<u>Example</u> $A = \mathcal{B}(H)$ $\quad H$ = fin-dim Hilb. space.

$a \in A \qquad a = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$ w.r.t. an orthonormal basis

$$\Rightarrow r_A(a) = \max_{1 \leq i \leq n} |\lambda_i| = \|a\|.$$

<u>Example</u> $A = \mathcal{B}(\mathbb{C}^2) \qquad a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$\Rightarrow r_A(a) = 0$, but $\|a\| > 0$.

<u>Exer.</u> $a \in A$ is nilpotent $\Rightarrow \sigma_A(a) = \{0\}$.

Thm. (Beurling, Gelfand)

A = unital Ban. alg, $a \in A \implies$

$$r(a) = \lim_{n\to\infty} \|a^n\|^{1/n} = \inf_{n\geq 1} \|a^n\|^{1/n}$$

Idea: $\leq$ triv;

$\geq$ $f \in A^*$; $\lambda \mapsto f((1-\lambda a)^{-1})$ $\quad f(a^n)$

Cor. $r(a) = 0 \iff \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}} = 0 \iff$

$\iff \forall \varepsilon > 0 \quad \|a^n\| = o(\varepsilon^n) \quad (n \to \infty)$

Def Such elements are called <u>quasinilpotent</u>.

Example/exer.

The Volterra integral operator

$$V_K : L^2[a,b] \to L^2[a,b], \quad (V_K f)(x) = \int_a^x K(x,y) f(y)\,dy$$

is quasinilp. $\forall$ bdd measurable K on $[a,b]\times[a,b]$.

Cor. 2. A = unital Ban. alg, $B \subset A$ closed subalg, $1_A \in B$. $\implies$

$\implies \forall b \in B \quad r_B(b) = r_A(b)$.

The maximal spectrum and the Gelfand transform

A = commutative unital algebra.

<u>Def.</u> An ideal $I \subsetneq A$ is <u>maximal</u> $\Longleftrightarrow$ $\nexists$ ideal J s.t. $I \subsetneq J \subsetneq A$.

<u>Exer.</u> I is maximal $\Leftrightarrow$ A/I is a field.

<u>Def.</u> The <u>maximal spectrum</u> of A is $\mathrm{Max}(A) = \{$maximal ideals of $A\}$.

<u>Exer/example</u>

$$\mathrm{Max}\,\mathbb{C}[t] \overset{1-1}{\rightleftarrows} \mathbb{C}$$

$$p \in \mathbb{C} \longmapsto m_p = \{f \in \mathbb{C}[t] : f(p) = 0\}.$$

<u>Prop.</u> Each proper ideal of A is contained in a max. ideal.

<u>Proof.</u> $I \subsetneq A$ ideal

$M = \{J : J \subsetneq A \text{ is an ideal}, I \subset J\}$

Claim: $(M, \subset)$ satisfies the conditions of Zorn's lemma.

Indeed: suppose $C \subset M$ is a chain.

Let $K = \bigcup\{J : J \in C\}$

K is an ideal, $I \subset K$.

$\forall J \in C \quad 1 \notin J \Rightarrow 1 \notin K \Rightarrow K \neq A$.

K is an upper bound for $C \Rightarrow$

$\Rightarrow M$ has a max. element. $\square$

<u>Def.</u> The <u>character space</u> of A is

$$\hat{A} = \{\chi : A \to \mathbb{C} : \chi \text{ is a character}, \chi \neq 0\}.$$

<u>Observe</u>: $\forall \chi \in \hat{A} \quad \operatorname{Ker}\chi \in \operatorname{Max}(A)$

<u>Prop</u> The map $\hat{A} \to \operatorname{Max}(A)$, $\chi \mapsto \operatorname{Ker}\chi$, is injective.

<u>Proof</u> $\chi_1, \chi_2 \in \hat{A}$; $\operatorname{Ker}\chi_1 = \operatorname{Ker}\chi_2 \Rightarrow$

$\Rightarrow \chi_1 = \lambda\chi_2 \quad (\lambda \in \mathbb{C})$; $\quad 1 = \chi_1(1) = \lambda\chi_2(1) = \lambda$

$\Rightarrow \chi_1 = \chi_2$. $\square$

<u>Example/exer.</u>

(1) $A = \mathbb{C}[t] \Rightarrow \hat{A} \to \operatorname{Max}(A)$ is a bijection.

(2) $A \supsetneq \mathbb{C}$ is a field $\Rightarrow \hat{A} = \emptyset$, but $\operatorname{Max}A = \{0\}$.

Lemma A = comm. unital Ban. alg $\Rightarrow$
$\Rightarrow$ each max. ideal of A is closed in A.

Proof Let $I \in \mathrm{Max}(A)$. $\Rightarrow$ $\overline{I}$ is an ideal
Suppose $I \neq \overline{I}$ $\Rightarrow$ $\overline{I} = A$ $\Rightarrow$
$\Rightarrow$ $\overline{I} \cap A^\times = \emptyset$ (because $A^\times$ is open in A)
$\Rightarrow$ $\overline{I} = A$, a contradiction $\square$

Cor. A comm. unital Ban. alg does not have dense proper ideals.

Thm. A = comm. unital Ban. alg $\Rightarrow$
$\Rightarrow$ the map $\hat{A} \to \mathrm{Max}(A)$, $\chi \mapsto \mathrm{Ker}\,\chi$,
is a bijection.

Observation: A = Ban. alg, $I \subset A$ closed 2-sided ideal of A $\Rightarrow$ A/I is a Ban. alg. w.r.t. $\|a + I\| = \inf\{\|a+b\| : b \in I\}$.

Proof of Thm Let $I \in \mathrm{Max}(A)$ $\Rightarrow$
$\Rightarrow$ A/I is a Ban. field $\Rightarrow$ $A/I \cong \mathbb{C}$

$$\begin{tikzcd} A \arrow[r, "\text{quot.}"] \arrow[rr, bend right, "\chi"'] & A/I \arrow[r, "\cong"] & \mathbb{C} \end{tikzcd}$$

$I = \mathrm{Ker}\,\chi$. $\square$

Cor. A = comm. unital Ban alg, $a \in A$.

$a \in A^\times \iff \forall \chi \in \hat{A} \quad \chi(a) \neq 0$.

Proof. $(\Rightarrow)$ clear.

$(\Leftarrow)$ Suppose $a \notin A^\times \Rightarrow Aa \subsetneq A \Rightarrow$
$\Rightarrow \exists\ I \in \mathrm{Max}(A)$ s.t. $I \supset Aa$; but $I = \mathrm{Ker}\,\chi$
$(\chi \in \hat{A}) \Rightarrow \chi(a) = 0$. $\square$

Convention Identify $\hat{A}$ with $\mathrm{Max}(A)$

Some facts on the weak* topology

E = normed space.

$\forall\, v \in E$ define a seminorm $\|\cdot\|_v$ on E^*
by $\|f\|_v = |f(v)|$.

Def. The weak* topology on E^* is the loc. convex topology gen. by $\{\|\cdot\|_v : v \in E\}$.

Explicitly: $\forall\, f \in E^*$ the standard subbase of nbhds of f (for wk*) is

$$\mathcal{G}_f = \{ U_{v,\varepsilon}(f) : v \in E,\ \varepsilon > 0 \},\ \text{where}$$

$$U_{v,\varepsilon}(f) = \{ g \in E^* : |g(v) - f(v)| < \varepsilon \}.$$

Facts: (0) wk* is Hausdorff.

(1) $(E^*, wk^*) \subset \mathbb{C}^E$

wk* = the restriction to E^* of the product (Tychonoff) topology on $\mathbb{C}^E$.

(2) $f_n \to f$ w.r.t wk* $\iff f_n(v) \to f(v)\ \forall v \in E$.

(3) $\forall\, v \in E$ let $\varepsilon_v : E^* \to \mathbb{C}$, $\varepsilon_v(f) = f(v)$.

wk* = the weakest topology on E that makes all ε_v continuous.

(4) X = top. space

A map $\varphi : X \to (E^*, wk^*)$ is cont $\iff$

$\iff \forall\, v \in E\ \ \varepsilon_v \circ \varphi : X \to \mathbb{C}$ is cont.

(5) $i_E : E \to E^{**}$ can. emb. $(v \mapsto \varepsilon_v)$.

$\operatorname{Im} i_E = \{\alpha \in E^{**} : \alpha \text{ is wk*-cont}\}$.

(6) E, F normed

A lin. oper. $T : (F^*, wk^*) \to (E^*, wk^*)$ is cont $\iff \exists$ a bdd lin. op. $S : E \to F$ st $S^* = T$.

(7) (Banach-Alaoglu Thm)

$\mathbb{B}_{E^*} = \{f \in E^* : \|f\| \leq 1\}$ is wk*-compact.

A = comm. unital Banach alg.

<u>Def</u> The <u>Gelfand topology</u> on $\mathrm{Max}(A) \cong \hat{A}$ is the restriction to $\hat{A}$ of the weak* top. on $\underline{A}^*$.

<u>Thm.</u> $\mathrm{Max}(A)$ is compact and Hausdorff.

<u>Proof</u> (A^*, wk^*) is Hausd $\Rightarrow$ so is $\hat{A}$.

$\hat{A} \subset \mathbb{B}_{A^*}$, $(\mathbb{B}_{A^*}, wk^*)$ is compact.

We have to show that $\hat{A} \subset \mathbb{B}_{A^*}$ is closed.

Let $a, b \in A$ Observe: the maps

$$\left.\begin{aligned} f \in A^* &\mapsto f(ab) - f(a)f(b) \\ f \in A^* &\mapsto f(1) \end{aligned}\right\} \text{ are cont wrt } wk^*$$

$$\hat{A} = \left\{ f \in A^* : \begin{array}{l} f(ab) - f(a)f(b) = 0 \ \ \forall a, b \in A; \\ f(1) = 1 \end{array} \right\}.$$

$\Rightarrow \hat{A}$ is closed in $\mathbb{B}_{A^*}$. $\square$

<u>Def</u> The <u>Gelfand transform</u> of $a \in A$ is

$\hat{a} : \mathrm{Max}(A) \to \mathbb{C}$, $\hat{a}(x) = x(a)$

<u>Prop</u> $\hat{a}$ is continuous.

<u>Proof</u> $\hat{a} = i_A(a)|_{\hat{A}}$; $i_A(a)$ is wk*-cont on A^*. $\square$

Def. The <u>Gelfand transform</u> of A is
$\Gamma_A : A \to C(\mathrm{Max}\,A)$, $a \in A \mapsto \hat{a}$.

Thm. (properties of Γ_A)
A = comm. unital Ban. algebra.
(1) Γ_A is a unital algebra homom.
(2) $\|\Gamma_A\| = 1$ (if $A \neq 0$)
(3) $\forall a \in A$ $\|\hat{a}\|_\infty = r_A(a)$
(4) $\forall a \in A$ $\sigma_A(a) = \hat{a}(\mathrm{Max}\,A)$.
(5) $\mathrm{Ker}\,\Gamma_A = \bigcap\{I : I \in \mathrm{Max}\,A\} =$
$= \{a \in A : a$ is quasinilp.$\}$.

Proof. (1) exer.
(4) We know: $\hat{a}(\mathrm{Max}A) = \sigma_{C(\mathrm{Max}\,A)}(\hat{a})$.
$\Longrightarrow$ it suff. to show that
$\Gamma(\text{noninv}) \subset$ noninv.
Suppose $a \notin A^\times \Rightarrow \exists\, \chi \in \hat{A}$ s.t. $\chi(a) = 0$,
that is, $\hat{a}(\chi) = 0$ $\Longrightarrow$ $\hat{a}$ is noninv in $C(\mathrm{Max}A)$
(3) follows from (4)
(2) $\forall a \in A$ $\|\hat{a}\|_\infty = r(a) \leq \|a\|$ $\Longrightarrow$ $\|\Gamma_A\| \leq 1$;

$\Gamma_A(1) = 1 \implies \|\Gamma_A\| = 1.$

(5) $\mathrm{Ker}\,\Gamma_A = \bigcap\{\mathrm{Ker}\,\chi : \chi \in \hat{A}\} =$

$= \bigcap\{I : I \in \mathrm{Max}(A)\} \underset{(3)}{=} \{\text{quasinilpotents}\}.$

$\square$

<u>Def.</u> A = unital comm. alg.

The <u>Jacobson radical</u> of A is

$J(A) = \bigcap\{I : I \in \mathrm{Max}(A)\}.$

A is <u>Jacobson semisimple</u> $\iff J(A) = 0$

<u>Cor.</u> $\mathrm{Im}\,\Gamma_A$ is spec. invariant in $C(\mathrm{Max}\,A)$.

<u>Proof.</u> $\Gamma(a) \in C(\mathrm{Max}\,A)^\times \implies a \in A^\times \implies$

$\implies \Gamma(a) \in (\mathrm{Im}\,\Gamma_A)^\times$ $\square$.