C^*-algebra: $(A, \|\cdot\|, *)$ $\quad a \in A \mapsto a^* \in A$

$\|a^* a\| = \|a\|^2 \qquad$ (C^*-axiom)

1943 Gelfand, Naimark

Quantum groups

$G = \text{group} \longmapsto A = $ Hopf algebra-like object $\xrightarrow[\text{(deformation)}]{}$ Quantum group

$\Delta : A \to A \otimes A$

Special cases

$G = $ semisimple Lie group $\mapsto \mathfrak{g} = \mathrm{Lie}(G) \mapsto U(\mathfrak{g}) \longmapsto U_q(\mathfrak{g})$, $U_h(\mathfrak{g})$

Drinfeld, Jimbo 1985/86

$$\boxed{G = \text{compact group} \mapsto C(G) \longmapsto C_q(G)}$$

Woronowicz, 1987

Relation:

$U_q(sl_2) \xmapsto[\text{(dual)}]{} \mathcal{O}_q(SL_2) \longmapsto \mathcal{O}_q(SU(2)) \xmapsto[C^*\text{-completion}]{} C_q(SU(2))$

A survey of Banach algebras

Convention: everything is over $\mathbb{C}$

Def Algebra = assoc $\mathbb{C}$-alg = assoc. ring A equipped with a $\mathbb{C}$-vec. space structure s.t. the mult. $A \times A \to A$ is $\mathbb{C}$-bilinear:

$$(\lambda a)b = a(\lambda b) = \lambda(ab) \quad (a, b \in A,\ \lambda \in \mathbb{C}).$$

A is unital $\iff \exists\ 1_A = 1 \in A : a \cdot 1 = 1 \cdot a = a\ \forall a$.

Def $\varphi : A \to B$ is an algebra homomorphism $\iff \varphi$ is a ring hom. and is $\mathbb{C}$-linear.
If A, B are unital, then φ is unital $\iff \varphi(1_A) = 1_B$.

Def. A normed algebra is an alg A equipped with a norm such that $\|ab\| \le \|a\|\|b\|$ $(a, b \in A)$ ($\|\cdot\|$ is submultiplicative).
If A is unital, then we require $\|1_A\| = 1$ (and $A \neq 0$).

Banach algebra = complete normed algebra

Exer. The mult. $A \times A \to A$ is cont.

<u>Example.</u> 0, $\mathbb{C}$

<u>Example.</u> X = a set

$\ell^\infty(X) = \{\text{bdd functions } X \to \mathbb{C}\}$

is a Banach alg wrt $\|f\|_\infty = \sup_{x \in X} |f(x)|$

<u>Example.</u> X = top. space.

$C_b(X) = C(X) \cap \ell^\infty(X) \subset \ell^\infty(X)$

closed subalg $\Longrightarrow$ a Banach alg.

<u>Def.</u> $f: X \to \mathbb{C}$ <u>vanishes at ∞</u> $\Longleftrightarrow \forall \varepsilon > 0$

$\exists$ a comp. subset $K \subset X$ s.t. $|f(x)| < \varepsilon \ \forall x \in X \setminus K$.

$\left(\lim_{x \to \infty} f(x) = 0 \right)$

<u>Example</u> $C_0(X) = \{f \in C(X) : f \text{ vanishes at } \infty\}$

$\subset C_b(X)$ is a closed ideal $\Rightarrow$ a Ban. alg.

<u>Special case</u>: X is compact $\Rightarrow$

$\Rightarrow C_0(X) = C_b(X) = C(X)$

<u>Example</u> (X, μ) = meas. space

$L^\infty(X, \mu)$ is a Ban. alg.

<u>Example / exer</u> $C^n[a,b]$ is a Ban. alg.

w.r.t. $\|f\| = \sum_{k=0}^{n} \frac{\|f^{(k)}\|_\infty}{k!}$

Example $K \subset \mathbb{C}$ compact

$\mathcal{A}(K) = \{ f \in C(K) : f \text{ is holom. on } \operatorname{Int} K \}$

$\subset C(K)$ closed subalg $\Rightarrow$ a Ban. alg.

$\overline{\mathbb{D}} = \{ z \in \mathbb{C} : |z| \le 1 \}$ $\quad \mathcal{A}(\overline{\mathbb{D}})$ is the disk algebra

Example E = Ban. space

$\mathcal{B}(E) = \{ \text{bdd linear operators on } E \}$.

is a Ban. alg.

Example $\mathcal{K}(E) = \{ \text{compact lin. operators on } E \}$

$\subset \mathcal{B}(E)$ closed 2-sided ideal. $\Rightarrow$ Ban. alg.

Example G - group

$\ell^1(G) = \{ f : G \to \mathbb{C} : \|f\|_1 = \sum_{g \in G} |f(g)| < \infty \}$.

Convolution: $f, g \in \ell^1(G)$ $\quad f * g : G \to \mathbb{C}$

$$(f*g)(x) = \sum_{y \in G} f(y) g(y^{-1}x) \qquad (x \in G)$$

Exer $(\ell^1(G), *)$ is a Banach alg.

A = unital algebra.

$A^\times = \{a \in A : a \text{ is invertible}\}$ multiplicative group of A.

<u>Def.</u> The <u>spectrum</u> of $a \in A$ is

$$\sigma_A(a) = \{\lambda \in \mathbb{C} : a - \lambda 1 \notin A^\times\}$$

<u>Example</u> $A = \mathbb{C}$ $\quad \sigma(\lambda) = \{\lambda\}$.

<u>Example</u> $A = \mathrm{End}_{\mathbb{C}}(E) = L(E)$

E = fin-dim. vec. space $\Longrightarrow \forall\, T \in L(E)$

$$\sigma_A(T) = \{\text{eigenvalues of } T\}$$

<u>Example.</u> X = set, $A = \mathbb{C}^X$ $\quad \forall f \in A$

$$\sigma_A(f) = f(X)$$

The same is true for $A = C(X)$ (X = top. space);

$A = C^\infty(M)$ (M = a manifold), ...

<u>Example</u> X = a set; $A = \ell^\infty(X)$. $\forall f \in A$

$$\sigma_A(f) = \overline{f(X)}.$$

<u>Def.</u> (X, μ) = meas space, $f : X \to \mathbb{C}$ measurable

$\lambda \in \mathbb{C}$ belongs to $\mathrm{ess.ran}(f)$ (<u>essential range</u>)

$\Longleftrightarrow \forall$ nbhd $U \ni \lambda$ $\quad \mu(f^{-1}(U)) > 0$

Example $A = L^\infty(X,\mu)$ $\quad \forall f \in A$

$\sigma_A(f) = \text{ess.ran}(f)$ (exer.)

Prop. A, B = unital alg, $\varphi : A \to B$ unital hom. Then

(1) $\varphi(A^\times) \subset B^\times$.

(2) $\forall a \in A \quad \sigma_B(\varphi(a)) \subset \sigma_A(a)$.

(3) $\varphi(A \setminus A^\times) \subset B \setminus B^\times \iff \forall a \in A \;\; \sigma_B(\varphi(a)) = \sigma_A(a)$.

Cor. A = unital alg, $B \subset A$ subalg, $1_A \in B$

$\Rightarrow \forall b \in B \quad \sigma_A(b) \subset \sigma_B(b)$.

Def. B is spectrally invariant in A $\iff$

$\iff \forall b \in B \quad \sigma_A(b) = \sigma_B(b) \iff A^\times \cap B = B^\times$.

Examples. (1) $C(X) \subset \mathbb{C}^X$ is spec. inv

(2) $\ell^\infty(X) \subset \mathbb{C}^X$ is not spec inv. $\forall$ inf. set X.

(3) $\mathcal{A}(K) \subset C(K)$ is spec. inv.

(4) $\mathcal{B}(E) \subset L(E)$ is spec. inv (E = Ban. space)

Thm (polynomial spectral mapping thm).

A = unital alg, $a \in A$, $f \in \mathbb{C}[t] \Longrightarrow$

$\sigma_A(f(a)) = f(\sigma_A(a))$ (unless $\sigma(a) = \emptyset$ and $f \in \mathbb{C}1$)

<u>Prop.</u> A = unital alg, $a \in A^\times \implies$
$\implies \sigma(a^{-1}) = \{\lambda^{-1} : \lambda \in \sigma(a)\}$.

<u>Thm.</u> A = unital Ban. alg. Then:

(1) $a \in A$, $\|a\| < 1 \implies 1 - a \in A^\times$, $(1-a)^{-1} = \sum_{n=0}^{\infty} a^n$.

(2) $A^\times$ is open in A.

(3) The map $A^\times \to A^\times$, $a \mapsto a^{-1}$, is continuous

<u>Def.</u> A = algebra. A <u>character</u> of A is an alg. hom. $A \longrightarrow \mathbb{C}$.

<u>Exer.</u> A is unital, $\chi : A \to \mathbb{C}$ character, $\chi \neq 0$
$\implies \chi(1) = 1$.

<u>Cor.</u> A = unital Ban. alg, $\chi : A \to \mathbb{C}$ a character
$\implies \chi$ is continuous, and $\|\chi\| \leq 1$.

<u>Proof.</u> We may assume that $\chi(1) = 1$.
If $\|\chi\| > 1$, or if χ is unbdd, then $\exists\, b \in A$ s.t. $|\chi(b)| > \|b\| \implies \exists\, a \in A$ s.t. $\|a\| < 1$, $\chi(a) = 1$. $\implies 1 - a \in A^\times \implies \chi(1-a) \neq 0$

$\chi(1-a) = 1 - \chi(a) = 0$, a contr. $\square$

Thm (Gelfand)

A = unital Ban. alg, $a \in A$. Then

(1) $\forall \lambda \in \sigma(a)$ $|\lambda| \leq \|a\|$.

(2) $\sigma(a)$ is compact

(3) If $A \neq 0$, then $\sigma(a) \neq \emptyset$.

Cor (Gelfand-Mazur thm)

A = Banach divison alg

(a unital Ban alg, $A \neq 0$, s.t. each $a \in A \setminus \{0\}$ is inv.)

$\Rightarrow A \cong \mathbb{C}$.

Proof. $\forall a \in A$ $\exists \lambda \in \mathbb{C}$ s.t. $a - \lambda 1 \notin A^\times \Rightarrow a = \lambda 1$. $\square$