Spectral theory of operators on a Hilbert space

This exercise sheet is intended for self-study only (that is, is plays no role in the grading). Nevertheless, it is strongly recommended to look at it before the exam.

4.1. Describe the polar decompositions (both "left" and "right") of the following operators:
(a) an orthogonal projection on a Hilbert space;
(b) the diagonal operator M_λ on ℓ^2 (where $\lambda \in \ell^\infty$);
(c) the multiplication operator M_φ on $L^2(X,\mu)$ (where (X,μ) is a measure space and $\varphi \in L^\infty(X,\mu)$);
(d) the shift operator on $\ell^2(\mathbb{Z})$;
(e) the right shift operator on ℓ^2;
(f) the left shift operator on ℓ^2.

4.2. Let T be a bounded linear operator on a Hilbert space. Can we always represent T in the form $T = US$, where S is a positive operator on H and U is a unitary operator on H?

4.3*. Let H be a Hilbert space. Prove that the group $\mathrm{U}(H)$ of unitary operators is a strong deformation retract of the group $\mathrm{GL}(H)$ of invertible operators.

Hint: use the polar decomposition and the continuity of $\sqrt{}$, exp, log on suitable subsets of $\mathscr{B}(H)$.

4.4. Let T be any of the following operators:
(a) an orthogonal projection on a Hilbert space;
(b) the diagonal operator M_λ on ℓ^2 (where $\lambda \in \ell^\infty$);
(c) the multiplication operator M_φ on $L^2(X,\mu)$ (where (X,μ) is a measure space, $\varphi \in L^\infty(X,\mu)$).
For each bounded Borel function $f\colon \sigma(T) \to \mathbb{C}$ define the operator $f(T)$ by an explicit formula.

4.5. Does the Borel functional calculus have **(a)** the spectral mapping property; **(b)** the superposition property?

4.6. Using the result of Exercise 4.4 (c) and the functional model of a normal operator, give another proof (different from that given at the lectures) of the existence of the Borel functional calculus.

Hint: you may use the fact that each σ-additive complex measure on a metrizable compact topological space is regular.

4.7. Show that for each unitary operator U on a Hilbert space H there exists a bounded selfadjoint operator T on H such that $U = \exp(iT)$ (compare with Exercise 3.14).

Hint: the function $[0, 2\pi) \to \mathbb{T}$, $t \mapsto \exp(it)$, is a Borel bijection.

4.8. Let H be a Hilbert space. Prove that the group $\mathrm{U}(H)$ of unitary operators and the group $\mathrm{GL}(H)$ of invertible operators are path connected.

Hint: see Exercises 4.3* and 4.7.

4.9. Show that a compact selfadjoint operator T is cyclic **(a)** if and **(b)** only if all the eigenvalues of T have multiplicity 1.

4.10. Let $\varphi\colon [a,b] \to \mathbb{R}$ be a strictly monotone, continuous function. Prove that the multiplication operator $M_\varphi\colon L^2[a,b] \to L^2[a,b]$ is cyclic.

4.11. Let T denote the operator on $L^2[0,1]$ defined by $(Tf)(t) = \sqrt{t}f(t)$. Find explicitly a positive Radon measure μ on $[0,1]$ and a unitary isomorphism $U\colon L^2[0,1] \to L^2([0,1],\mu)$ which establishes a unitary equivalence between T and the multiplication operator M_t given by $(M_t f)(t) = tf(t)$.

4.12. Let T be a cyclic selfadjoint operator on a Hilbert space H. Prove that an operator $S \in \mathscr{B}(H)$ commutes with T if and only if there exists a bounded Borel function $f\colon \sigma(T) \to \mathbb{C}$ such that $S = f(T)$.

4.13*. Let $\varphi\colon [a,b] \to \mathbb{R}$ be a bounded Borel function. Suppose that the multiplication operator $M_\varphi\colon L^2[a,b] \to L^2[a,b]$ is cyclic. Prove that there exists a measure zero set $A \subset [a,b]$ such that φ is injective on $[a,b] \setminus A$[1].

4.14. (a) By using the spectral theorem, prove that a bounded normal operator P such that $\sigma(P) \subset \{0,1\}$ is a projection.
(b) Prove (a) without using the spectral theorem.

4.15. (a)-(c) Describe the spectral measures of the operators from Exercise 4.4.

4.16*. Let H be an infinite-dimensional separable Hilbert space. Prove that $\mathscr{K}(H)$ is a unique closed two-sided ideal of $\mathscr{B}(H)$ different from 0 and $\mathscr{B}(H)$.

Hint. Let $0 \neq I \subset \mathscr{B}(H)$ be a two-sided ideal. Recall the standard proof of the simplicity of the matrix algebra $M_n(\mathbb{C})$, and apply the same argument to show that I contains all finite rank operators. If I contains at least one noncompact operator, apply the spectral theorem.

[1] The converse is also true, but this involves some nontrivial facts in measure theory.