Topological vector spaces

Exercises marked by "**-B**" are optional. If you solve such exercises, you will earn bonus points.

2.1. Let X and Y be topological vector spaces. Show that
(a) a linear operator $X \to Y$ is continuous iff it is continuous at 0;
(b) the set $\mathscr{L}(X,Y)$ of continuous linear operators from X to Y is a vector subspace of the space of all linear operators from X to Y.

2.2. Is there at least one continuous norm on the following topological vector spaces?
(a) $\mathbb{K}^X$ (where X is a set); **(b)** $C(X)$ (where X is a metrizable topological space);
(c) the space $\mathscr{O}(U)$ of holomorphic functions on an open set $U \subset \mathbb{C}$ (we equip $\mathscr{O}(U)$ with the topology induced from $C(U)$);
(d) $C^\infty[a,b]$; **(e)** $C^\infty(U)$, where $U \subset \mathbb{R}^n$ is an open set; **(f)** $\mathscr{S}(\mathbb{R}^n)$.

2.3. Let X be a Hausdorff locally convex space, and let P be a defining family of seminorms on X. Show that X is normable iff P is equivalent to a finite subfamily $P_0 \subset P$.

2.4. (a)-(f) Which spaces of Exercise 2.2 are normable?

2.5. Let X be a Hausdorff locally convex space, and let P be a defining family of seminorms on X. Show that X is metrizable iff P is equivalent to an at most countable subfamily $P_0 \subset P$.

Hint. If $(p_n)_{n\in\mathbb{N}}$ is a sequence of seminorms, then the function

$$\rho(x,y) = \sum_n \frac{1}{2^n} \min\{p_n(x-y), 1\} \quad \text{or, if you like,} \quad \rho(x,y) = \sum_n \frac{1}{2^n} \frac{p_n(x-y)}{1+p_n(x-y)}$$

satisfies the triangle inequality.

2.6. (a)-(f) Which spaces of Exercise 2.2 are metrizable?

2.7. Let X and Y be normed spaces. Prove that the topology on $\mathscr{B}(X,Y)$ generated by the operator norm is stronger (that is, finer) than the strong operator topology, and that the strong operator topology is stronger than the weak operator topology.

2.8. Let T_ℓ and T_r denote the left shift and the right shift operators on ℓ^2. Do the sequences $(T_\ell^n)_{n\in\mathbb{N}}$ and $(T_r^n)_{n\in\mathbb{N}}$ converge **(a)** w.r.t. the norm topology on $\mathscr{B}(\ell^2)$; **(b)** w.r.t. the strong operator topology on $\mathscr{B}(\ell^2)$; **(c)** w.r.t. the weak operator topology on $\mathscr{B}(\ell^2)$?

2.9-B. Let X be a finite-dimensional vector space. Show that there is only one topology on X making X into a Hausdorff locally convex space.

2.10. Let X be a set. Prove that for each $f \in \mathbb{K}^X$ the multiplication operator $M_f \colon \mathbb{K}^X \to \mathbb{K}^X$, $M_f(g) = fg$, is continuous.

2.11. Let $U \subset \mathbb{R}^n$ be an open set. Prove that each linear differential operator $\sum_{|\alpha|\leqslant N} a_\alpha D^\alpha$ on $C^\infty(U)$ (where $a_\alpha \in C^\infty(U)$) is continuous.

2.12. Let $\mathbb{D}_R = \{z \in \mathbb{C} : |z| < R\}$. Given $f \in \mathscr{O}(\mathbb{D}_R)$, let $c_n(f) = f^{(n)}(0)/n!$. Show that the topology of compact convergence on $\mathscr{O}(\mathbb{D}_R)$ is generated by the family $\{\|\cdot\|_{r,\infty} : 0 < r < R\}$ of seminorms, where $\|f\|_{r,\infty} = \sup_{n\geqslant 0} |c_n(f)| r^n$.

2.13. Recall (see the lectures) that the standard topology on the Schwartz space $\mathscr{S}(\mathbb{R})$ is given by the family $\{\|\cdot\|_{k,\ell} : k,\ell \in \mathbb{Z}_{\geqslant 0}\}$ of seminorms, where $\|\varphi\|_{k,\ell} = \sup_{x\in\mathbb{R}} |x^k \varphi^{(\ell)}(x)|$. Show that the following families of seminorms generate the same topology on $\mathscr{S}(\mathbb{R})$:
(a) $\{\|\cdot\|_\infty^{(p)} : p \in \mathbb{Z}_{\geqslant 0}\}$, where $\|\varphi\|_\infty^{(p)} = \sup_{k\leqslant p,\, x\in\mathbb{R}} (1+x^2)^{p/2} |\varphi^{(k)}(x)|$;
(b) $\{\|\cdot\|_1^{(p)} : p \in \mathbb{Z}_{\geqslant 0}\}$, where $\|\varphi\|_1^{(p)} = \max_{k\leqslant p} \int_{\mathbb{R}} (1+x^2)^{p/2} |\varphi^{(k)}(x)|\, dx$.

2.14-B. The space $s(\mathbb{Z})$ of *rapidly decreasing sequences* is defined as follows:

$$s(\mathbb{Z}) = \Big\{ x = (x_n) \in \mathbb{K}^{\mathbb{Z}} : \|x\|_k = \sum_{n\in\mathbb{Z}} |x_n||n|^k < \infty \ \forall k \in \mathbb{Z}_{\geqslant 0} \Big\}.$$

The standard topology on $s(\mathbb{Z})$ is determined by the family $\{\|\cdot\|_k : k \in \mathbb{Z}_{\geqslant 0}\}$ of norms. Topologize the space $C^\infty(S^1)$ (by analogy with $C^\infty[a,b]$), and construct a topological isomorphism $C^\infty(S^1) \cong s(\mathbb{Z})$. (*Hint:* the isomorphism takes each $f \in C^\infty(S^1)$ to the sequence of its Fourier coefficients.)

2.15-B. Let (X,μ) be a finite measure space, and let $L^0(X,\mu)$ denote the space of equivalence classes of measurable functions on X (two functions are equivalent if they are equal μ-almost everywhere). For each $f, g \in L^0(X,\mu)$ we let

$$\rho(f,g) = \int_X \min\{|f-g|, 1\}\, d\mu.$$

Prove that
(a) ρ is a metric making $L^0(X,\mu)$ into a topological vector space;
(b) a sequence of measurable functions converges in $L^0(X,\mu)$ iff it converges in measure;
(c) $(L^0[0,1])^* = 0$.

2.16. Let $\langle X, Y\rangle$ be a dual pair of vector spaces. Show that
(a) $\dim X < \infty \iff \dim Y < \infty \iff$ the weak topology $\sigma(X,Y)$ is normable;
(b) the weak topology $\sigma(X,Y)$ is metrizable $\iff$ the dimension of Y is at most countable;
(c) the weak topology on an infinite-dimensional normed space and the weak* topology on the dual of an infinite-dimensional Banach space are not metrizable.

2.17. Describe all continuous linear functionals on $\mathbb{K}^X$ (where X is a set), and show that the weak topology on $\mathbb{K}^X$ is identical to the original topology.

2.18. Let $e_n = (0,\ldots,0,1,0,\ldots)$, where 1 is in the nth slot. Does (e_n) converge weakly in c_0 and in ℓ^p $(1 \leqslant p < \infty)$?

2.19. Give an example of a discontinuous linear operator T between Hausdorff locally convex spaces X and Y such that T is continuous w.r.t. the weak topologies on X and Y.

2.20. **(a)** Give an example of a Banach space X and a norm closed vector subspace $Y \subset X^*$ that is not weakly* closed.
(b) Show that, if X is nonreflexive, then X^* contains a subspace Y satisfying (a).

2.21-B. Show that each weakly convergent sequence in ℓ^1 is norm convergent.