Functional Analysis (Operator Theory)

Exam 29.05.2020

1. **Commutative Banach algebras**
   - The maximal spectrum and the character space of a commutative unital algebra.
   - The closedness of maximal ideals of a commutative unital Banach algebra.
   - The 1-1 correspondence between the character space and the maximal spectrum of a commutative unital Banach algebra.
   - The invertibility criterion in terms of characters.
   - The Gelfand topology on the maximal spectrum.
   - The compactness of the maximal spectrum (unital case).
   - The Gelfand transform of a commutative unital Banach algebra.
   - Properties of the Gelfand transform.
   - The maximal spectrum and the Gelfand transform for subalgebras of $C(X)$.
2. **C^*-algebras and continuous functional calculus**
   - Banach $*$-algebras. C^*-algebras. Examples.
   - Selfadjoint elements, unitary elements, normal elements.
   - The main property of the spectral radius of a normal element. Corollaries: the uniqueness of the C^*-norm on a $*$-algebra; the automatic continuity of $*$-homomorphisms.
   - The main property of the spectrum of a selfadjoint element in a C^*-algebra. Corollaries: characters of a C^*-algebra are $*$-characters; the spectral invariance of C^*-subalgebras.
   - The 1st (commutative) Gelfand-Naimark theorem (unital case).
   - The continuous functional calculus in C^*-algebras (definition, a necessary condition for the existence, the existence theorem).
   - The spectral mapping theorem and the superposition property for the functional calculus.
   - The semicontinuity of the spectrum in Banach algebras.
   - The joint continuity of the functional calculus in C^*-algebras.
   - Positive elements of C^*-algebras.
   - Properties of the set of positive elements.
   - Square roots.
   - Decomposing selfadjoint elements as differences of positive elements.
   - Kaplansky's theorem ($x^*x \geqslant 0$). Characterizations of positivity.
3. **Positive operators and polar decomposition**
   - Sesquilinear forms. Polarization.
   - Hermitian forms and their characterization.
   - The sesquilinear and the quadratic form associated to an operator between inner product spaces.

- Characterizations of selfadjoint and positive operators in terms of quadratic forms.
- Characterizations of isometries, coisometries, unitary operators, and orthogonal projections in algebraic terms.
- Partial isometries: several equivalent definitions.
- The polar decompositions ("left" and "right") of bounded linear operators between Hilbert spaces.
- The uniqueness of the polar decompositions.

4. **Spectral Theorem I: functional models**

   - $*$-representations and $*$-modules. Cyclic $*$-modules. Examples.
   - The functional model for a cyclic $*$-module over $C(X)$.
   - The correspondence between normal operators and $*$-modules over $C(K)$ ($K \subset \mathbb{C}$).
   - The correspondence between $C(K)$-$*$-module morphisms and intertwining operators.
   - $*$-cyclic operators and their relation to cyclic $*$-modules.
   - The functional model for a $*$-cyclic normal operator.
   - Hilbert direct sums of Hilbert spaces and of $*$-modules.
   - A decomposition of a $*$-module into a Hilbert direct sum of cyclic submodules.
   - The functional model for a $*$-module over $C(X)$.
   - The functional model for a normal operator (Spectral Theorem I).

5. **Spectral Theorem II: Borel functional calculus**

   - The correspondence between bounded linear operators on a Hilbert space and bounded sesquilinear forms.
   - The weak measure topology and the weak operator topology.
   - The WMT-density of $C(X)$ in $B(X)$.
   - The separate continuity of the multiplication in $(B(X), \mathrm{WMT})$ and in $(\mathscr{B}(H), \mathrm{WOT})$.
   - The Borel functional calculus for a normal operator (definition, uniqueness, a necessary condition for the existence).
   - The canonical extension of representations of $C(X)$ to $B(X)$.
   - The existence of a Borel functional calculus for a normal operator (Spectral Theorem II).

6. **Spectral Theorem III: spectral measures**

   - Finitely additive spectral measures and associated complex measures. Examples.
   - Integration of bounded measurable functions with respect to a finitely additive spectral measure. The 1-1 correspondence between finitely additive spectral measures and representations of $B_{\mathscr{A}}(X)$.
   - Regular spectral measures on a compact Hausdorff topological space.
   - The 1-1 correspondence between regular spectral measures and WMT-WOT-continuous representations of $B(X)$.
   - The characterization of representations of $C(X)$ in terms of regular spectral measures.
   - The spectral decomposition of a normal operator (Spectral Theorem III).