Functional Analysis (Operator Theory)

Midterm exam 13.03.2020

1. Fredholm operators. The Fredholm index. Basic examples. The additivity of the index.

2. Kato's lemma (the image of a Fredholm operator is closed). The Closed Image Theorem for operators between Banach spaces. A corollary: the Fredholmness and the index of the dual operator.

3. The ascent and the descent of a linear operator. Riesz's theorem on operators "$\mathbf{1}$+compact". The Fredholm alternative. Abstract Fredholm theorems in Schauder's form.

4. Properties of the spectrum of a compact operator on a Banach space.

5. Topological direct sums and complemented subspaces of normed spaces. Characterizations of complemented subspaces (in terms of projections and in terms of quotients). Finite-dimensional subspaces and closed subspaces of finite codimension are complemented.

6. The Nikolski–Atkinson criterion for Fredholm operators. The Calkin algebra. The essential spectrum of a linear operator. The compactness and the nonemptiness of the essential spectrum.

7. The set of Fredholm operators is open, and the index is locally constant.

8. The stability of the index under compact perturbations. Nikolski's characterization of Fredholm operators of index zero.

9. Toeplitz operators on the Hardy space. The index formula for a Toeplitz operator with continuous symbol.

10. Topological vector spaces. The topology generated by a family of seminorms. The continuity of algebraic operations, a characterization of convergence in terms of seminorms, a criterion of the Hausdorff property. Examples: spaces of continuous and smooth functions, the Schwartz space, the strong operator topology, the weak operator topology.

11. Convex, circled, and absolutely convex sets in vector spaces. Convex, circled, and absolutely convex hulls. Absorbing sets. The Minkowski functional and its properties. Locally convex spaces. The "polynormability" of locally convex spaces.

12. The continuity criterion for a seminorm on a locally convex space. The continuity criterion for a linear operator between locally convex spaces. The domination relation for families of seminorms. Equivalent families of seminorms. Examples.

13. Linear functionals on locally convex spaces (extension from subspaces, separation of points, separation of points and subspaces).

14. Dual pairs of vector spaces. The weak topology of a dual pair. Special cases and basic properties of the weak topology. A characterization of linear functionals continuous with respect to the weak topology. A characterization of reflexive Banach spaces in terms of topologies on the dual.

15. Dual operators between dual pairs of vector spaces. An existence criterion for the dual operator in terms of weak continuity. Relations between the continuity, the weak continuity, and the existence of the dual for an operator between locally convex spaces (in particular, between normed spaces).
16. Annihilators and polars. Basic properties of annihilators and polars. The bipolar theorem (for dual pairs and for locally convex spaces). Corollaries: the closure of an absolutely convex set equals the weak closure; the double annihilator theorem; a density criterion for a vector subspace; relations between kernels and images of a linear operator and of the dual operator. Goldstine's theorem.
17. Equicontinuous families of linear operators. The Banach–Alaoglu–Bourbaki theorem. A characterization of reflexive Banach spaces in terms of weak compactness.