This is NOT an exercise sheet, but is just a selection of exercises meant to be discussed at the blackboard during a "traditional" seminar.

Embeddings and quotients

Let X and Y be normed spaces. Recall that a linear operator $T\colon X \to Y$ is a *coisometry* if it takes the open unit ball of X onto the open unit ball of Y.

Exercise 1. Let X be a normed space, and let $f\colon X \to \mathbb{K}$ be a linear functional.
(a) Show that f is open (unless $f = 0$).
(b) Show that f is a coisometry iff $\|f\| = 1$.
Warning: do not forget about the case where $\mathbb{K} = \mathbb{C}$.

Exercise 2. Let X and Y be normed spaces, and let $T\colon X \to Y$ be a linear operator.
(a) Prove that if T maps the closed unit ball of X onto the closed unit ball of Y, then T is a coisometry.
(b) Is the converse true?
(c) Show that T is an injective coisometry iff T is an isometric isomorphism.

Exercise 3. Let $\alpha \in \ell^\infty$, and let X denote either ℓ^p or c_0. Let M_α be the *diagonal operator* on X defined by $M_\alpha(x) = (\alpha_i x_i)_{i\in\mathbb{N}}$. Find a condition on α that is necessary and sufficient for M_α to be **(a)** topologically injective; **(b)** open; **(c)** an isometry; **(d)** a coisometry.

Exercise 4. Let (Ω, μ) be a σ-finite measure space, and let f be a bounded measurable function on Ω. Answer questions **(a)** – **(d)** of the previous exercise for the *multiplication operator* M_f on $L^p(\Omega, \mu)$, $M_f(g) = fg$.

Exercise 5. Let X be a normed space, and let $X_0 \subset X$ be a vector subspace. Prove that
(a) the quotient seminorm on X/X_0 is indeed a seminorm;
(b) the topology on X/X_0 determined by the quotient seminorm is the quotient topology (i.e., a subset $U \subset X/X_0$ is open iff $Q^{-1}(U)$ is open in X, where $Q\colon X \to X/X_0$ is the quotient map).

Definition 1. Let X be a normed space. A series $\sum_{n=1}^\infty x_n$ in X is *absolutely convergent* if the series $\sum_{n=1}^\infty \|x_n\|$ converges in $\mathbb{R}$.

Exercise 6. Show that a normed space X is complete iff each absolutely convergent series in X converges. (We have used this fact at the lecture.)

Exercise 7. Let (Ω, μ) be a measure space, and let $B(\Omega)$ denote the space of bounded measurable functions on Ω. We equip $B(\Omega)$ with the sup norm. Construct an isometric isomorphism between $L^\infty(\Omega, \mu)$ and a quotient of $B(\Omega)$. Deduce that $L^\infty(\Omega, \mu)$ is complete.

Exercise 8. Construct **(a)** a topological isomorphism between c_0 and a quotient of $C[0, 1]$; **(b)** an isometric isomorphism between ℓ^1 and a quotient of $L^1[0, 1]$.